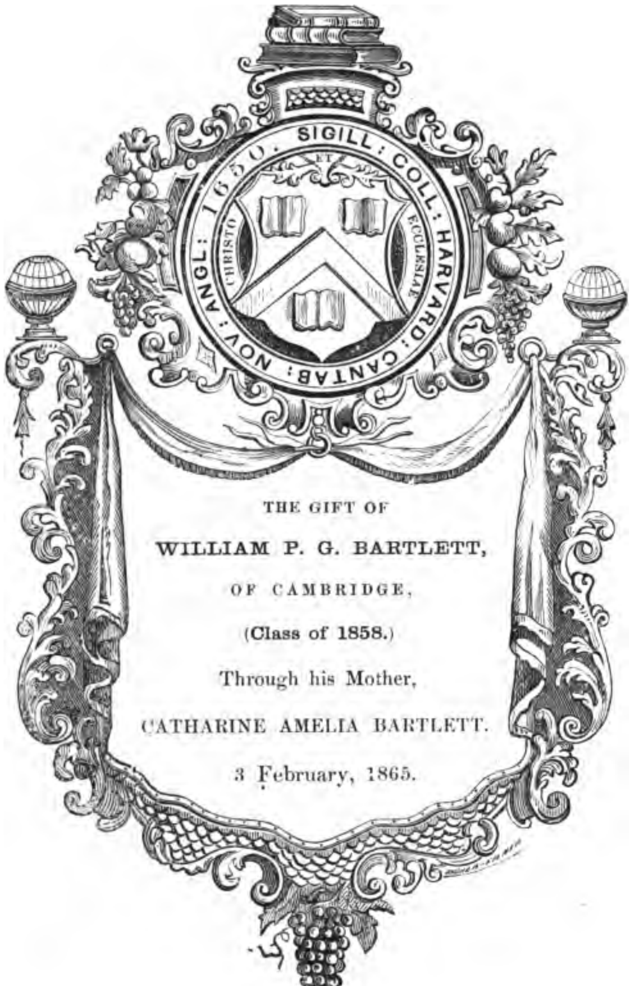

SIGILL : COLL : HARVARD : CANTAB : NOV : ANGL : 1650.
CHRISTO ET ECCLESIAE

THE THEORY

OF

ELLIPTIC INTEGRALS,

&c. &c.

S. C. Whitbread Esq F.R.S
from the author

LONDON:
SPOTTISWOODES and SHAW,
New-street-Square.

THE THEORY

OF

ELLIPTIC INTEGRALS,

AND THE

PROPERTIES OF SURFACES OF THE SECOND ORDER,

APPLIED TO THE INVESTIGATION OF

THE MOTION OF A BODY

ROUND A FIXED POINT.

BY

JAMES BOOTH, LL.D., F.R.S., &c.

CHAPLAIN TO THE MOST HONOURABLE THE MARQUESS OF LANSDOWNE,
AND FORMERLY PRINCIPAL OF BRISTOL COLLEGE.

LONDON:
GEORGE BELL, FLEET STREET.
CAMBRIDGE: JOHN DEIGHTON. DUBLIN: HODGES AND SMITH.
1851.

" Quant aux sciences des phénomènes naturels, nous ne doutons point que les surfaces du second degré ne doivent s'y présenter aussi dans un grand nombre de questions, et y jouer un rôle aussi important que celui des sections coniques dans le système planétaire." — CHASLES, *Aperçu Historique*, p. 251.

TO

THE MOST HONOURABLE

THE MARQUIS OF LANSDOWNE, K.G.

LORD PRESIDENT OF THE COUNCIL,

THE FOLLOWING TREATISE

IS RESPECTFULLY INSCRIBED

BY

THE AUTHOR.

PREFACE.

THE investigations given in the following pages were made, the greater portion of them, several years ago. Some of them appeared from time to time in those periodical publications whose pages are open to discussions on subjects of this nature.

In this treatise a complete investigation has been attempted of the laws of the motion of a rigid body round a fixed point, free from the action of accelerating forces, based on the properties of surfaces of the second order, of the curves in which these surfaces intersect, and on the theory of elliptic integrals. The results which have been obtained are exact and not approximate, general and not restricted by any imposed hypothesis.

That the theory of the rotation of a rigid body round a fixed point might be made to rest on the properties of the ellipsoid, was long ago shown by Legendre, and more recently by Poinsot in his brief but elegant tract, the "*Théorie nouvelle de la Rotation des Corps.*" Professor De Morgan very justly observes, in his great work on the Differential and Integral Calculus, "that the long, isolated, and inelegant investigations which usually fill up the chapters of works on dynamics which treat of rotatory motions might be almost entirely avoided, if the student were supposed to have that knowledge of the ellipsoid which he is supposed to have of the ellipse before he reads on the theory of gravitation." The ultimate analysis, however, or the dynamical solution of this problem, must be sought in the evaluation of those mathematical expressions known as elliptic integrals. At this point writers usually have abandoned the subject, or confined themselves to the discussion of particular hypotheses, and the deduction of approximate results.

In connection with this portion of the subject, two curves, which have not, within my knowledge, been noticed by geometers, will claim the reader's attention. I have named them the *spherical parabola*, and the *logarithmic ellipse*. The former may be traced on the surface of a sphere, the latter on a paraboloid of revolution. These curves are of importance. They are the geometrical representatives of elliptic integrals of the first order, and of the logarithmic form of the third order. They complete the geometrical expressions for those integrals. The *four* forms may be represented by four ellipses, one *plane*, two *spherical*, and one *parabolic*. The formulæ of comparison for those integrals given by Legendre and by others follow as simple geometrical conclusions from the properties of these curves. As an example, may be mentioned, Lagrange's scale of modular transformations in the first order, and the linear logarithmic and circular residuals which present themselves in the comparison of the different orders of elliptic integrals.

I have carefully abstained from introducing any methods which, to one moderately versed in the first principles of the integral calculus, might not fairly be assumed as known. There is one exception. In a few cases, where the method was peculiarly applicable, I have ventured to make use of a new kind of coordinates, which were named in a short tract, published some years ago, *tangential coordinates*. The reader may, however, if he chooses, omit those applications, without breaking the continuity of the subject.

I have not been led away by mathematical pedantry to attempt to render this essay purely algebraical, by rejecting geometrical conceptions and the aids thence derived to simplicity and clearness; knowing that, very often, the elegance of the analysis is owing to the distinctness of the graphical conception, and that though the forms of the reasoning may be different, the subject matter is identically the same.

J. B.

May 1*st*, 1851.

CONTENTS.

SECTION I.

SECTION II.

SECTION III.

SECTION IV.

SECTION V.

SECTION VI.

SECTION VII.

SECTION VIII.

SECTION IX.

ERRATA.

Page 17, formula (15), for "$\int$" read "$\frac{\tan\beta}{\tan\alpha}\cos\alpha\int$."

19, top line, for "sin" read "$\sin\beta$."

24, line 19, for "$\frac{d\sigma}{d\lambda}$" read "$\frac{d\sigma}{d\lambda}$."

27, line 2, for "$\cos\sigma$" read "$\cos\alpha$."

29, line 8, for "(A^2-B^2)" read "$(A^2-B^2)^2$."

30, line 20, for "Lionville" read "Liouville."

32, line 9, for "d" read "$d\mu$."

34, line 23, omit "*the halves of.*"

41, line 7, for "a" read "$\sqrt{a}$."

96, last line, for "is constant" read "varies as the time."

ON

ROTATORY MOTION.

INTRODUCTION.

The problem of the rotation of a rigid body round a fixed point is one that has engaged the attention of the most eminent mathematicians of Europe. A century has passed away since D'Alembert first, and Euler soon after, investigated the analytical conditions of such a motion, and formed those differential equations, on the integration of which, the determination of the motion ultimately depends. In their investigations, which were purely analytical, they used to a great extent the principle of the transformation of co-ordinates; a method of research, it must be conceded, which, though unexceptionable on the ground of mathematical rigour, is sometimes found to lead through cumbrous processes to complicated results.

Some years afterwards, Lagrange took up the subject, and developed the theory in formulæ of great symmetry and generality. Combining the principle of D'Alembert with that of virtual velocities, he deduced the differential equations of motion, containing six quantities to be determined. By means of these equations, the three components of the angular velocities round the principal axes, which determine the position of the instantaneous axis of rotation in the body, and the three other angular quantities which define the position of the body itself in space, at any epoch, may be expressed in terms of the time. But those quantities, however they may be determined, furnish us, as it has justly been observed, with no conception of

the motion during the time. They prove to us that the body, after the lapse of a certain time, must be in a certain position; but we are not shown how it gets there. We may, by means of calculations, more or less long and complicated, ascertain the bearings of the body at any required instant. We cannot, so to speak, accompany it during its motion. It is determined *per saltum,* and not continuously. We are wholly unable to keep it in view and follow it, as it were, with our eyes during the whole progress of rotation.

To furnish a clear idea of the rotatory motion of a body round a fixed point, free from the action of accelerating or other external forces, but in motion from the influence of one or more primitive impulses, was the object of a memoir, presented several years ago to the Institute, by that eminent mathematician M. Poinsot. In this memoir, still unpublished, but of which the author himself at the time printed and published a very luminous *précis,* the motion of a body round a fixed point, and free from the action of accelerating forces, is reduced to the motion of a certain ellipsoid, whose centre is fixed, and which rolls, without sliding, on a plane fixed in space. The axes of this ellipsoid are assumed proportional to the inverse square roots of the moments of inertia round the principal axes of the body, passing through the fixed point, and coinciding in direction with them. He states as his final result, that the time and the other ultimate quantities must be determined by the aid of elliptic functions. He does not, however, give any account of the processes by which he arrived at his results, and few of the attempts which have since been made to supply that omission have been very successful.

Nearly about the same time, the late Professor M'Cullagh turned his attention to this problem, which had then attracted considerable notice, owing to the recent researches of Poinsot. He adopted an ellipsoid, the reciprocal of that chosen by the latter geometer, and deduced those results which long before had been arrived at by the more operose methods of Euler and Lagrange. His method of investigation was, however, peculiarly his own. In his hands, as in those of Newton, the method of geometrical infinitesimals was fitted to grapple with difficulties which ordinary minds can barely overcome, with all the aids

which a refined analysis is so well calculated to bestow. Professor M'Cullagh was, however, so much occupied with other and perhaps more important subjects of physical inquiry, that he never took occasion to publish the results of his researches on rotatory motion. However simple and elegant his methods may have been, they would not harmonise with the modes of investigation adopted in the following pages.

The idea of substituting, as a means of investigation, an ideal ellipsoid, having certain relations with the actually revolving body, claims the illustrious Legendre as its author. Although he conducts his own investigations on principles altogether different, he yet seems well aware of the use which might be made of this happy conception. In his *Traité des Fonctions Elliptiques*, he says:

"Quelles que soient la figure et la constitution intérieure d'un corps solide qui peut librement tourner dans tous les sens autour d'un point fixe, et qui n'est soumis à l'action d'aucune force accélératrice, le mouvement de ce corps peut toujours être assimilé à celui d'un ellipsoide homogène de même masse, dont les demi-axes principaux du corps proposé ont les mêmes moments d'inertie, et qui aurait reçu la même vitesse initiale dans le même sens et autour du même axe de rotation.

"En effet, les seuls élémens qui, dans la théorie précédente dependent de la figure du corps et de la loi qui suit la densité de ses différentes molécules, sont les quantités A, B, C, par lesquelles se forment les momens d' inertie du corps relativement aux troix axes principaux. Donc si ces quantités sont égales dans deux corps, et si l'impulsion primitive est la même ces deux corps auront nécessairement la même position et les mêmes vitesses au bout d'un temps quelconque."—*Traité des Fonctions Elliptiques*, tom. i. p. 410.

Several years ago, when engaged in applying a new analytical method, founded on a peculiar system of co-ordinates, which he named *tangential coordinates*, the author was led to views somewhat similar, from remarking the close analogy, or rather identity, which exists between the formulæ for finding the position of the principal axes of a body, and those for determining the symmetrical diameters of an ellipsoid. He still further observed, that the expression for the length of a perpendicular from the

centre on a tangent plane to an ellipsoid, in terms of the cosines of the angles which it makes with the axes, is precisely the same in form with that which gives the value of the moment of inertia round a line passing through the origin. Guided by this analogy he was led to assume an ellipsoid, the squares of whose axes should be *directly* proportional to the moments of inertia round the coinciding principal axes of the body.

At first sight the inverse ellipsoid, assumed by Poinsot, may seem to possess some advantages over the direct ellipsoid, at least so far as such an ellipsoid may be said to approximate in form to the natural body. For example, if we consider the case of the rotation of a solid homogeneous ellipsoid round its centre, the ideal or mathematical ellipsoid of Poinsot will bear a resemblance to the figure actually in motion. In the direct ellipsoid of moments, which is made the instrument of investigation in the following pages, this resemblance does not exist, for the coinciding axes of the material and mathematical ellipsoids are such, that the sum of their squares is constant. Should the revolving figure be an oblate spheroid, its mathematical representative will be a prolate spheroid. The reader must bear this diversity of figure in mind, in applying the conclusions of theory to an actually revolving ellipsoid. Although it may seem a matter of little moment, which of the ellipsoids we choose as the geometrical substitute for the revolving body, it is not so in reality, when we come to treat of the properties of the integrals which determine the motion. These integrals depend on the properties of those curves of double flexion, in which cones of the second degree are intersected by concentric spheres. By means of the properties of these curves, a complete solution may be obtained, even in the most general case to which only an approximation has hitherto been made. The solution of a mathematical problem may only then be said to be complete, when in the final result the calculation of the sought quantities may be made to depend on some known elementary quantity or quantities, such as a certain right line, an arc of a circle, &c. So in this problem, the elliptic transcendents, to the determination of which the calculation of the motion is ultimately reduced, are shown to represent arcs of spherical conic sections, whose elements depend on the nature of the body, and on the magni-

tude and position of the impressed moment. In all the solutions of this problem which have hitherto appeared, the investigations are brought to a close, when the expressions, either for the time or other sought quantity, are reduced so as to include the square roots of quadrinomials involving the independent variable to the fourth power. In this treatise the investigations are continued beyond this point, and a brief account of what the author has been enabled to accomplish in this portion of the subject may not be out of place.

It is a well-known theorem, established by Legendre, that two circular elliptic functions of the third order, having the same modulus and amplitude, but with positive and negative parameters respectively, may be connected by a certain equation which he establishes. This formula may be found in any treatise on elliptic functions. It is shown in the following pages that these circular forms arise from treating the element of the spherical conic, either as the hypothenuse of an infinitesimal right-angled spherical triangle, or as a portion of a circle having the same curvature. If we adopt the former principle, we shall obtain for the arc an elliptic function of the third order, circular form, and negative parameter. Selecting the latter, we shall get a circular form of the same order, with a positive parameter. The modulus and amplitudes are shown to be the same for both. Equating these dissimilar expressions for the same arc, we obtain Legendre's theorem.

Again, in prosecuting these researches, the author believes he has been the first to hit upon a theorem which gives the true representative curve of elliptic functions of the first order. The true geometrical representative of this function has long been felt as a desideratum in this department of mathematical science. Legendre, and others since his time have devised curves whose rectification may be effected by elliptic functions of the first order. But they are barren of analogies and complex in their construction: they are deficient in that simplicity which one seldom fails to meet with, when a question is contemplated from its true point of view. Moreover, the properties of those curves throw but little light upon, neither do they give any geometrical explanation of, the most elementary properties of these functions. The theorem is as follows:

Any spherical conic section, the tangents of whose principal semiarcs may be constituted the coordinates of an equilateral hyperbola, whose transverse semiaxis is 1, *may be rectified by an elliptic function of the first order.* As a corollary from this,

The quadrature of any spherical conic section, the cotangents of whose principal semiarcs may be constituted the ordinates of an equilateral hyperbola, whose transverse axe is 1, *may be effected by an elliptic function of the first order.*

Hence it follows, that there is a particular class of elliptic functions of the third order, which may be reduced to elliptic functions of the first order, a theorem that has long been known.

This particular species of spherical ellipse, being the gnomonic projection, on the surface of a sphere, of a plane parabola touching it at its focus, the author has named the *spherical parabola;* a name to which its close analogies to the plane curve give it a clear title.

As in this particular species of spherical ellipse we may take either the focus or the centre as the origin of the spherical radii vectores, in effecting the processes of rectification, we are unexpectedly presented with Lagrange's scale of modular transformations, as also with the other equally well-known theorem by which the successive amplitudes are connected. In fact these theorems are simple corollaries from the geometrical properties of those curves.

An elliptic function of the first order being shown to be only a particular case of elliptic functions of the third order, as the circle is a species of ellipse, it follows that the analogies between functions of the first and third orders will be more numerous and intimate than between the second and either of the others. Such is in fact the case. Elliptic functions of the first and third orders constantly occur in combination. In the discussions of the following pages, for example, functions of the first and third orders present themselves in various combination, while a function of the second order does not once occur in the essay. The spherical parabola developes many curious and peculiar properties, some of which are established in the following pages.

The application of the theory of elliptic functions to the discussion of the problem of a rigid body revolving round a fixed point, has led to the remarkable theorem, that, —

The length of the spiral between two of its successive apsides, described in absolute space, on the surface of a fixed concentric sphere, by the instantaneous axis of rotation, is equal to a quadrant of the spherical ellipse described by the same axis on an equal sphere, moving with the body.

The ordinary equations of motion being established, the author proceeds to show that if the direct ellipsoid of moments be constructed, the rotatory motion of a body, acted on solely by primitive impulses, may be represented by this ellipsoid moving round its centre, in such a way that its surface shall always pass through a point fixed in space. This point, so fixed, is the extremity of the axis of the plane of the impressed couple, or of the plane known to mathematicians as the invariable plane of the motion.

But a still clearer idea of the motion of such a body may be formed by the aid of another theorem, which shows that the whole motion of a revolving body may be represented by a cone which rolls, without sliding, on a fixed plane passing through its vertex, while this plane revolves with an uniform motion round its own axis. This, perhaps, is the simplest conception we can form of a revolving body. Now the principal axes of this cone, and its focal lines, depend on the constitution and form of the body, or in other words are functions of the moments of inertia round the principal axes; while the initial position of the plane of the impressed couple in the body will determine the tangent plane to this cone. But when the two focal lines of a cone, and a tangent plane to it, are given, the cone may as completely be determined as a conic section when its foci and a tangent to it are given. Nothing more simple than this conception, a cone rigidly connected with the body, the position of whose focal lines, and whose principal vertical angles, depend on the form and constitution of the body, revolves without sliding on a plane, while the plane itself revolves uniformly round its own axis. We may also observe, that when the plane of the impressed couple passes through one of the focals of the rolling cone, the motion is *sui generis;* it no longer may be represented by a rolling cone. The cone degenerates into a plane segment of a circle, which swings round one or other of

the cyclic axes of the ellipsoid of moments, these cyclic axes being the boundaries of the circular segment.

Although it may be, and doubtless is, very satisfactory in this way to be enabled to place before our eyes, so to speak, the very actual motion of the revolving body, yet it is not on such grounds solely that the following essay has been published. Were the theory of no other use than to give strength and clearness to vague and obscure notions on this confessedly most difficult subject, enough had been already accomplished by the celebrated geometer whose name is so deservedly associated with this subject. It is as a method of investigation that it must rest its claims to the notice of mathematicians; as a means of giving simple and elegant interpretations of those definite integrals, on the evaluation of which the dynamical state of a body at any epoch can alone be ascertained. If the author has to any degree succeeded in accomplishing this, it is because he has drawn largely upon the properties of lines and surfaces of the second order, and of those curve lines in which these surfaces intersect. If he has been enabled to advance any thing new, it is owing solely to the somewhat unfrequented path he has pursued. That it was antecedently probable such might lead to undiscovered truths, no one conversant with the applications of mathematical conceptions to the discussions of those sciences will deny. To introduce auxiliary surfaces into the discussions and investigations of physical science is an idea no less luminous than it has been successful. The properties of such surfaces often aid our conceptions, or facilitate our calculations in a very remarkable manner. M. Dupin, for example, reduces the problem of the equilibrium of a floating body to that of a solid resting on a horizontal plane; the solid being the envelope of all the planes which retrench from the floating body a given volume. We have a still more striking instance in the wave-theory of light. We there find the surface of elasticity the equimomental surface in the theory of rotation. Few indeed there are among mathematicians who require to be informed how the wave-surface of Fresnel, or its polar reciprocal, the surface of wave-slowness of Sir William Hamilton, have served to clear our conceptions on a subject as yet scarcely understood, to realize and embody an indistinct and shadowy

theory. Nay, more, the geometrical properties of the surface of *wave-slowness* in the hands of Sir W. Hamilton have led to the anticipation of the theory of conical refraction. They have enabled us to invert the natural order of induction and to place theory in advance of experiment. Were further illustration needed, one might refer with confidence to the treatise of Mac Laurin on the figure of the earth; to the researches of Clairault on the same subject, to the investigations of Poisson, Chasles, and Ivory, on the attraction of ellipsoids. A theorem in surfaces of the second order, on which he has bestowed his name, enabled the latter to evade the difficulties of the problem on which he was engaged. So true is the fine anticipation of Bacon:—"For as Physicall knowledge daily growes up, and new Actioms of nature are disclosed; there will be a necessity of new Mathematique inventions."*

The author has taken occasion, during the progress of the essay, to derive those partial solutions on particular hypotheses, which are given in the usual text-books on this portion of dynamical science. To the reader familiar with those solutions it will, doubtless, be satisfactory to see them follow, as simple conclusions, from principles more widely general. These partial solutions serve, as it were, to test the truth and accuracy of the principles on which the entire theory is based. To those who read the subject as a portion of academical study, it will, no doubt, prove interesting to discover an additional link connecting the deductions of pure thought with the laws of matter and motion. They will not fail to observe the analogy, that as the properties of the sections of a cone by a plane have elucidated the motions of translation of the planets in their orbits, so likewise the theory of the rotation of those bodies, round their axes, may be founded on the properties of the sections of a cone by a sphere.

* Of the Advancement of Learning, book iii. chap. 6.

SECTION I.

I. As the properties of cones of the second degree, and of the curves of double flexion, in which their surfaces may be intersected by concentric spheres, will in the following pages continually be referred to, it is proper to establish such of them here as we shall have occasion to make use of. Some of them we believe will not be found in any published treatise on the subject.

Definition.

A spherical conic may be defined as the curve of intersection of a cone of the second degree with a concentric sphere.

When we consider only the closed curve through the centre of which the real axe of the cone passes, the curve may be named the spherical ellipse.

In the spherical ellipse there are two points analogous to the foci of the plane ellipse, such that the sum of the arcs of the great circles drawn from those points to any point on the curve is constant.

Let α and β be the principal semiangles of the cone; 2α and 2β are therefore the *principal arcs* of the spherical ellipse. Let two right lines be drawn from the vertex of the cone in the plane of the angle 2α, making with the internal axe of the cone angles ε, such that $\cos\varepsilon = \frac{\cos\alpha}{\cos\beta}$, these lines are termed *focals* or the *focal lines* of the cone. The points in which they meet the surface of the sphere are termed the foci of the spherical ellipse.

It is generally known that every umbilical surface of the second order has two concentric circular sections, which, in the case of cones, pass through the greater of the external axes. Perpendiculars to the planes of those sections passing through the vertex—they may be termed the *cyclic axes* — make with the

internal axe of the cone, in the plane of 2β — the plane passing through the internal and the least external axe — angles η, such that

$$\cos\eta = \frac{\sin\beta}{\sin\alpha} \quad . \quad . \quad . \quad (1).$$

Let a series of planes be drawn through the vertex and perpendicular to the successive sides of the cone, this series of planes will envelope a second cone, which is usually termed the *supplemental cone* to the former. The cones are so related that the planes of the circular sections of the one are perpendicular to the focals of the other, and reciprocally.

II. The equation of the spherical ellipse may be found as follows, from simple elementary considerations.

Let 2α and 2β be the greatest and least vertical angles of the cone; the origin of coordinates being placed at the common centre of the sphere and cone. Let the internal axe of the cone meet the surface of the sphere in Z, which point may be taken as the pole. The greater external axe, or the angle 2α being taken in the plane of yz, the lesser angle 2β will be in the plane of xz. Let ρ be an arc of a great circle drawn from the point Z to any point Q on the curve. ψ being the angle which the plane of this circle makes with the plane of 2α, then we shall have for the polar equation of the spherical ellipse,

$$\frac{\cos^2\psi}{\tan^2\alpha} + \frac{\sin^2\psi}{\tan^2\beta} = \frac{1}{\tan^2\rho}.$$

To show this, through the point Z let a tangent plane be drawn to the sphere intersecting the cone in an ellipse, which for perspicuity may be named the *plane base* of the cone, in contradistinction to the portion of the sphere within the cone which may be designated the *spherical* base of the cone. The great circle passing through Z and Q will cut the plane base of the cone in the radius vector R, and if we write A and B for semiaxes of this section, we shall have the following equation, being the common polar equation of the ellipse, the centre being the pole, $\frac{1}{R^2} = \frac{\cos^2\psi}{A^2} + \frac{\sin^2\psi}{B^2}$.

Now the radius of the sphere being k, and ρ, α, β, the angles

subtended at the centre by R, A, B; $R = k \tan \rho$, $A = k \tan \alpha$, $B = k \tan \beta$, whence

$$\frac{1}{\tan^2 \rho} = \frac{\cos^2 \psi}{\tan^2 \alpha} + \frac{\sin^2 \psi}{\tan^2 \beta} \quad . \quad . \quad (2).$$

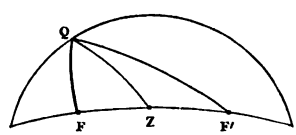
We may write this equation in the following form

$$\frac{1 - \sin^2 \rho}{\sin^2 \rho} = \frac{\cos^2 \psi}{\sin^2 \alpha}(1 - \sin^2 \alpha) + \frac{\sin^2 \psi}{\sin^2 \beta}(1 - \sin^2 \beta)$$

or reducing $$\frac{1}{\sin^2 \rho} = \frac{\cos^2 \psi}{\sin^2 \alpha} + \frac{\sin^2 \psi}{\sin^2 \beta} \quad . \quad . \quad (3).$$

This is the equation of the spherical ellipse in another form, which may be obtained independently, by orthogonally projecting the spherical ellipse on the plane of the external axes.

III. If in the major principal arc 2α of the spherical ellipse we assume two points equidistant from the centre, the distance ϵ being determined by the condition $\cos \epsilon = \frac{\cos \alpha}{\cos \beta}$, the sum of the arcs of the great circles drawn from those points to any point of the spherical ellipse is constant, and equal to the principal arc 2α. That this value of $\cos \epsilon$ is correctly assumed will be apparent from the consideration, that should such a property exist generally, it must hold for the extremity of the principal minor arc. Now in this case α, β, ϵ, are the three sides of a right-angled spherical triangle; whence $\cos \alpha = \cos \beta \cos \epsilon$.



Let θ and θ' denote the arcs drawn from the points F, F′ to the point Q upon the curve, $QZ = \rho$, and the angle QZF $= \psi$, $FZ = F'Z = \epsilon$.

Then as FZQ, F′ZQ are spherical triangles, we get

$$\cos \psi = \frac{\cos \theta - \cos \epsilon \cos \rho}{\sin \epsilon \sin \rho} \text{ (a)}, \quad - \cos \psi = \frac{\cos \theta' - \cos \epsilon \cos \rho}{\sin \epsilon \sin \rho} \text{ (b)},$$

$\cos \epsilon = \frac{\cos \alpha}{\cos \beta}$ (c), and the equation of the curve given in (2)

$$\cot^2 \rho = \cot^2 \alpha \cos^2 \psi + \cot^2 \beta \sin^2 \psi \quad . \quad \text{(d)}.$$

Between (a) (b) (c) (d) we must eliminate ρ ψ and ϵ. Adding together (a) and (b); also subtracting (b) from (a), we get $\cos \theta + \cos \theta' = 2 \cos \rho \cos \epsilon$; and $\cos \theta - \cos \theta' = 2 \sin \rho \sin \epsilon \cos \psi$;

from (d) $1 = \cot^2 \alpha \tan^2 \rho \cos^2 \psi + \tan^2 \rho \cot^2 \beta - \tan^2 \rho \cot^2 \beta \cos^2 \psi$;

or $\left(\frac{\cos^2 \beta - \cos^2 \alpha}{\sin^2 \alpha \ \sin^2 \beta}\right) \sin^2 \rho \cos^2 \psi = \cot^2 \beta - \frac{\cos^2 \rho}{\sin^2 \beta}$; substituting for $\sin \rho \cos \psi$, its value deduced by subtracting (b) from (a), we find $\cos^2 \alpha \ (\cos \theta - \cos \theta')^2 + \sin^2 \alpha \ (\cos \theta + \cos \theta')^2 = \sin^2 2\alpha$, or $\cos^2 \theta + \cos^2 \theta' - 2 \cos \theta \cos \theta (\cos^2 \alpha - \sin^2 \alpha) = 1 - \cos^2 2\alpha$;

whence $\cos^2 2\alpha - 2 \cos \theta \cos \theta' \cos 2\alpha = 1 - \cos^2 \theta - \cos^2 \theta'$.

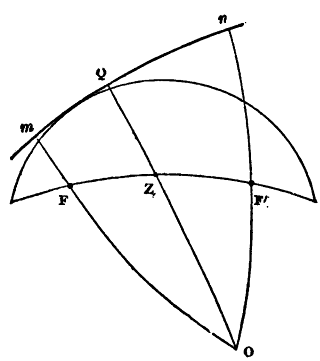
Completing the square and reducing, we obtain

$\cos 2\alpha = \cos \theta \cos \theta' \mp \sin \theta \sin \theta' = \cos (\theta \pm \theta')$ or

$$2\alpha = \theta \pm \theta' \quad . \quad . \quad . \quad (4).$$

The positive sign to be taken when the curve is the spherical ellipse.

IV. The product of the sines of the perpendicular arcs let fall from the foci, on the arc of a great circle which touches the spherical ellipse, is constant.



Let $\varpi, \varpi', \varpi''$, be the perpendicular arcs, from the centre and two foci, on the tangent arc mn. These three arcs will meet in the point o the pole of the arc mn. Let p be the perpendicular from the centre on the right line which touches the plane elliptic base; of this right line, $m\,n$ is the projection. We shall therefore have

$$p^2 = A^2 \cos^2 \lambda + B^2 \sin^2 \lambda,$$

or $\tan^2 \varpi = \tan^2 \alpha \cos^2 \lambda + \tan^2 \beta \sin^2 \lambda$,

whence $\cos^2 \varpi = \frac{\cos^2 \alpha}{1 - \sin^2 \varepsilon \sin^2 \lambda}$.

Now $\text{FZQ} = \lambda$, whence in the spherical triangle FZO

as $\text{FO} = \frac{\pi}{2} - \varpi'$, $\text{ZO} = \frac{\pi}{2} - \varpi$, we shall have

$\cos \lambda = \frac{\sin \varpi' - \cos \varepsilon \sin \varpi}{\sin \varepsilon \cos \varpi}$. In the other spherical triangle F′ZO we shall also have $-\cos \lambda = \frac{\sin \varpi'' - \cos \varepsilon \sin \varpi}{\sin \varepsilon \cos \varpi}$.

Adding first, and then subtracting those equations, one from the other, we find

$$\sin \varpi' + \sin \varpi'' = 2 \cos \varepsilon \sin \varpi,$$
$$\sin \varpi' - \sin \varpi'' = 2 \sin \varepsilon \cos \varpi \cos \lambda$$

Squaring those equations, and subtracting the latter from the former, we shall obtain

$$\sin \varpi' \sin \varpi'' = \cos^2 \varepsilon - \cos^2 \varpi \,(1 - \sin^2 \varepsilon \sin^2 \lambda).$$

Substituting for $\cos \varpi$ its value given above, and reducing,

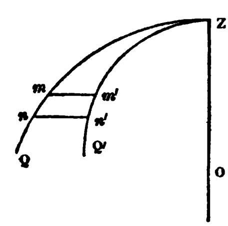
$$\sin \varpi' \sin \varpi'' = \sin (\alpha + \varepsilon) \sin (\alpha - \varepsilon) \quad . \quad . \quad (5).$$

V. The area of any portion of a spherical surface bounded by a closed curve, may be determined by the formula,

$$\text{area} = \int_0^{2\pi} d\psi \int_0^{\rho} d\sigma \,[\, \sin \sigma \,],$$

where σ is the arc of a great circle intercepted between the fixed point Z taken within the curve as pole, and any variable point m assumed within the bounding curve on the surface of the sphere; ρ being the spherical radius vector of the curve measured from the pole Z, and passing through m; while ψ is the angle which the plane of the great circle, passing through the points Z, m, makes with the fixed plane of a great circle passing through Z.

Let O be the centre of the sphere, Z the pole, m the assumed point, Z Q the great circle passing through them. Through Z let a great circle O Z Q′ be drawn, indefinitely near to the former, $d\psi$ being the angle between the planes. Through m let a plane be drawn perpendicular to the axis O Z, meeting the great circle O Z Q′ in m'. Through n a point on Z Q indefinitely near to m a parallel plane being drawn, it will meet the great circle O Z Q′ in a point n', indefinitely near to m'.



Now it is manifest from this construction that the whole spherical area to be determined is the sum of all the indefinitely small trapezia, such as $m n m' n'$, into which it may in this manner be divided. To compute the value of this elementary trapezium, we have $m m'$

$= \sin \sigma \, d\psi$, $mn = d\sigma$. As the pole Z is within the curve, the limits of σ are 0 and ρ; and as the surface is assumed to extend all round Z, the limits of ψ are 0 and 2π. Whence,

$$\text{area} = \int_0^{2\pi} d\psi \int_0^{\rho} d\sigma \, [\sin \sigma].$$

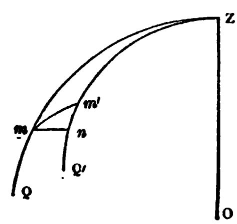
Integrating this equation between the limits 0 and ρ, we find

$$\text{area} = \int_0^{2\pi} d\psi \, [1 - \cos \rho] \quad . \quad . \quad (6).*$$

The second integration can be accomplished only when we know the relation between ρ and ψ, or the equation of the bounding curve.

VI. To find an expression for the length of a curve described on the surface of a sphere.

Let m and m' be two consecutive points on the curve, ZQ, ZQ′ two great circles passing through them, inclined to each other at the angle $d\psi$. Through m let a plane be drawn perpendicular to OZ, and meeting the great circle ZQ′ in n. Then, ultimately mnm' may be taken as a right-angled triangle, whence $\overline{mm'}^2 = \overline{mn}^2 + \overline{m'n}^2$. Now, $mm' = d\sigma$, $mn = \sin \rho \, d\psi$, $Zm = \rho$, $m'n = d\rho$. Whence $d\sigma = [d\rho^2 + \sin^2 \rho \, d\psi^2]^{\frac{1}{2}}$.



Integrating this expression between the limits $\rho_{\prime}$, $\rho_{\prime\prime}$ or ψ and 0, accordingly as we take ρ or ψ for the independent variable, we obtain

$$\text{arc} = \int_{\rho_{\prime}}^{\rho_{\prime\prime}} d\rho \left[1 + \sin^2 \rho \, \frac{d\psi^2}{d\rho^2}\right]^{\frac{1}{2}}; \text{ or arc} = \int_0^{\psi} d\psi \left[\frac{d\rho^2}{d\psi^2} + \sin^2 \rho\right]^{\frac{1}{2}} \quad (7).$$

* Equation (6) may be established by the help of the simplest elementary principles. We know that the surface of the segment of a sphere comprised between a tangent plane and a parallel secant plane is equal to the circumference of a great circle multiplied into the distance between these planes. This distance is $1 - \cos \rho$; ρ being the arc of a great circle, measured from the point of contact of the tangent plane to the parallel secant plane. If through the diameter perpendicular to these planes we draw two great circles, inclined, one to the other, at the angle $d\psi$, the surface of the spherical wedge thus formed will be

$$d\psi(1 - \cos \rho).$$

VII. To find the area of the spherical ellipse.

Resuming equations (2) and (3) of the spherical ellipse,

$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}, \text{ and } \frac{1}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}+\frac{\sin^2\psi}{\sin^2\beta}.$$

Dividing the former by the latter, and reducing, we find

$$\cos\rho=\cos\alpha\frac{\sqrt{1+\frac{\tan^2\alpha}{\tan^2\beta}\tan^2\psi}}{\sqrt{1+\frac{\sin^2\alpha}{\sin^2\beta}\tan^2\psi}} \quad . \quad . \quad (8).$$

Substituting this value of $\cos\rho$ in the general expression for the spherical area (6), we obtain the result

$$\text{area}=\psi-\cos\alpha\int d\psi\left[\frac{1+\frac{\tan^2\alpha}{\tan^2\beta}\tan^2\psi}{1+\frac{\sin^2\alpha}{\sin^2\beta}\tan^2\psi}\right]^{\frac{1}{2}} \quad . \ (9).$$

To integrate this equation, let us make the assumption

$$\tan\psi=\frac{\tan\beta}{\tan\alpha}\tan\phi \quad . \quad . \quad . \quad (10);$$

and we shall find, on making the necessary transformations, the area $=\psi$

$$-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int d\phi\left[\frac{1}{\left\{1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\phi\right\}\sqrt{1-\left(\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\beta}\right)\sin^2\phi}}\right](11).$$

Let A and B be the semiaxes of the plane elliptic base of the cone, and e its eccentricity, then we shall obviously have

$$e^2=\frac{A^2-B^2}{A^2}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha} \quad . \quad . \quad . \quad . \quad (12);$$

and ε being the angle between the spherical focus and centre,

$$\cos\varepsilon=\frac{\cos\alpha}{\cos\beta} \text{ (Art. III.), whence } \sin^2\varepsilon=\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\beta} (13).$$

Introducing those relations into (11), we obtain the formula

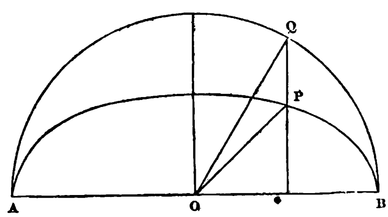
$$\text{area}=\psi-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int d\phi\left[\frac{1}{[1-e^2\sin^2\phi]\sqrt{1-\sin^2\varepsilon\sin^2\phi}}\right] (14).$$

This is an elliptic function of the *third order* and *circular* form, since e^2 is less than 1 and greater than $\sin^2 \varepsilon$.

This seems to be the simplest form to which the quadrature of the spherical ellipse can be reduced. The *parameter* and squared *modulus* of the elliptic transcendent being the squares of the eccentricities of the plane and spherical ellipses respectively.

We shall show hereafter that there is a class of spherical ellipses whose quadrature may be effected by elliptic functions of the *first* order.

VIII. To determine the geometrical signification of the angle of reduction φ, in the above transformation.



On the major axis of the plane elliptic base of the cone, let a semicircle be described. Let OP be drawn, making the angle ψ with the major axis OB. Let the ordinate through P be produced to meet the circle in Q, join OQ;

then $\dfrac{\tan \psi}{\tan \text{QOB}} = \dfrac{\text{PD}}{\text{QD}} = \dfrac{\text{B}}{\text{A}} = \dfrac{\tan \beta}{\tan \alpha}$, but $\dfrac{\tan \psi}{\tan \varphi} = \dfrac{\tan \beta}{\tan \alpha}$; see (10);

whence $\text{QOB} = \varphi$, or φ is the *eccentric anomaly* of the point P.

Now, when $\psi = 0$, $\varphi = 0$, and when $\psi = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ whence φ and ψ coincide at those limits. Writing S for the area of the quadrant of the spherical ellipse; as the surface evidently consists of four symmetrical quadrants, the area or length of one quadrant will manifestly be one-fourth of the area or length of the whole; whence

$$\text{S} = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} d\varphi \left[\frac{1}{[1 - e^2 \sin^2 \varphi] \sqrt{1 - \sin^2 \varepsilon \sin^2 \varphi}}\right] \quad . \quad . \ (15).$$

IX. To find the length of an arc of the spherical ellipse.

In this case it will be found simpler to integrate the differential expression for an arc of the curve, with respect to ρ, rather than ψ the independent variable in the last problem.

Resuming the equation established in (7),

$$\text{arc} = \int d\rho \left[1 + \left(\sin\rho \frac{d\psi}{d\rho}\right)^2\right]^{\frac{1}{2}},$$ we deduce from

(3), $$\sin^2\psi = \frac{\sin^2\beta}{\sin^2\rho}\left\{\frac{\sin^2\alpha - \sin^2\rho}{\sin^2\alpha - \sin^2\beta}\right\},\quad \cos^2\psi = \frac{\sin^2\alpha}{\sin^2\rho}\left\{\frac{\sin^2\rho - \sin^2\beta}{\sin^2\alpha - \sin^2\beta}\right\} \quad (16);$$

differentiating the former with respect to ψ and ρ, and eliminating $\sin\psi$, $\cos\psi$; using for this purpose the relations established in (16),

$$\frac{d\psi}{d\rho} = \frac{-\sin\alpha\sin\beta\cos\rho}{\sin\rho\sqrt{\sin^2\alpha - \sin^2\rho}\sqrt{\sin^2\rho - \sin^2\beta}} \quad . \quad . \quad (17).$$

Substituting this value of $\frac{d\psi}{d\rho}$ in the general expression for the arc, the resulting equation will be found,

$$\text{arc} = \int d\rho \left[\frac{\sin\rho\sqrt{\cos^2\rho - \cos^2\alpha\cos^2\beta}}{\sqrt{\sin^2\alpha - \sin^2\rho}\sqrt{\sin^2\rho - \sin^2\beta}}\right] \quad . \quad . \quad (18).$$

An elliptic function which may be reduced to the usual form by the help of the following transformation:

$$\text{Assume } \cos^2\rho = \frac{\sin^2\alpha\cos^2\phi + \sin^2\beta\sin^2\phi}{\tan^2\alpha\cos^2\phi + \tan^2\beta\sin^2\phi} \quad . \quad . \quad (19).$$

The limits of integration being 0 and $\frac{\pi}{2}$. Differentiating (19), introducing into (18) the relations assumed in (19), we find the arc =

$$\frac{\tan\beta}{\tan\alpha}\sin\beta\int d\phi\left[\frac{1}{\left\{1 - \left(\frac{\tan^2\alpha - \tan^2\beta}{\tan^2\alpha}\right)\sin^2\phi\right\}\sqrt{1 - \left(\frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}\right)\sin^2\phi}}\right] (20).$$

We have shown in (12) that $e^2 = \frac{\tan^2\alpha - \tan^2\beta}{\tan^2\alpha}$, and in Sec. I., that $\cos\eta = \frac{\sin\beta}{\sin\alpha}$, whence $\sin^2\eta = \frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}$ (21).

Integrating between the limits 0 and $\frac{\pi}{2}$, and writing Σ for the length of a quadrant of the spherical ellipse,

$$\Sigma = \frac{\tan\beta}{\tan\alpha}\sin\int_0^{\frac{\pi}{2}} d\phi\left[\frac{1}{\{1-e^2\sin^2\varphi\}\sqrt{1-\sin^2\eta\sin^2\varphi}}\right] \quad (22).$$

An elliptic function, which is also of the third order and circular form, since e^2 is greater than $\sin^2\eta$ and less than 1.

This is the simplest form to which the rectification of an arc of a spherical ellipse can be reduced. The parameter and squared modulus of the elliptic function of the third order being the squares of the eccentricity of the plane elliptic base, and of the sine of half the angle between the circular sections of the cone.

We shall show presently that there is a class of spherical ellipses whose rectification may be effected by an elliptic function of the first order.

It is easily shown that the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ of the elliptic function in (22) is the square root of the expression which may be termed the *criterion* of *circularity*. As this coefficient is manifestly real, the expression itself must be positive, or the elliptic function is of the circular form.

X. Let α' and β' be the principal arcs of the supplemental cone, α' being in the plane of β, and β' in that of α. Let Σ' be the length of a quadrant of the spherical ellipse the intersection of this cone with the concentric sphere. Then we deduce from (22)

$$\Sigma' = \frac{\tan\beta'}{\tan\alpha'}\sin\beta'\int^{\frac{\pi}{2}} d\varphi\left[\frac{1}{\{1-e'^2\sin^2\varphi\}\sqrt{1-\sin^2\eta'\sin^2\varphi}}\right]$$

Now as the cones are supplemental,

$\alpha+\beta'=\frac{\pi}{2}$, $\beta+\alpha'=\frac{\pi}{2}$, whence $\sin\alpha'=\cos\beta$, $\sin\beta'=\cos\alpha$,

$\cos\alpha'=\sin\beta$, $\cos\beta'=\sin\alpha$, therefore $\frac{\tan\beta'}{\tan\alpha'}=\frac{\tan\beta}{\tan\alpha}$,

$e'^2=e^2$, and $\sin\eta'=\sin\varepsilon$ . . . . . . (23).

Introducing these transformations into the last formula

$$\Sigma' = \frac{\tan\beta}{\tan\alpha}\cos\alpha\int_0^{\frac{\pi}{2}} d\varphi\left[\frac{1}{\{1-e^2\sin^2\varphi\}\sqrt{1-\sin^2\varepsilon\sin^2\varphi}}\right] \quad (24).$$

Now if we turn to the expression found for the area of a spherical ellipse and given in (14), we shall find that it consists of two parts, a circular arc, and an elliptic integral identically the same with the one just investigated, when taken between the limits 0 and $\frac{\pi}{2}$. We thus arrive at the very remarkable result, that the rectification of a spherical ellipse depends on the quadrature of the supplemental ellipse, and reciprocally.*

If we add together (14) and (24),

$$\mathrm{S} + \Sigma' = \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (25);$$

or taking the whole surface $4\,\mathrm{S}$ of the spherical conic, and the circumference $4\,\Sigma'$ of the supplemental conic, introducing, moreover, k the radius of the sphere, we obtain the remarkable theorem

$$4\,\mathrm{S} + 4\,k\Sigma' = 2\,k^2\pi \quad . \quad . \quad . \quad (26).$$

Now $4\,k\,\Sigma'$ is twice the lateral surface of the supplemental cone, and $4\,\mathrm{S}$ is the surface of the spherical ellipse. We may therefore infer that

The spherical base of any cone, together with twice the lateral surface of the supplemental cone, is equal to the surface of the hemisphere.

XI. Let $4\,\mathrm{S}'$ denote the spherical base of the supplemental cone, and L the lateral surface of the original cone: from the preceding equations we obtain

$$2\,\mathrm{S} + \mathrm{L}' = k^2\pi,\ 2\,\mathrm{S}' + \mathrm{L} = k^2\pi \quad . \quad . \quad (27).$$

Adding those equations,

$$4\,(\mathrm{S} + \mathrm{S}') + 2\,(\mathrm{L} + \mathrm{L}') = 4k^2\pi \quad . \quad (28).$$

Subtracting one from the other,

$$4\,(\mathrm{S} - \mathrm{S}') = 2\,(\mathrm{L} - \mathrm{L}') \quad . \quad . \quad (29);$$

* The discovery of this remarkable relation between the length and area of a spherical ellipse is due to the late Professor MacCullagh, to whom mathematical science is indebted for some new and beautiful theorems in this department of geometrical research.

or, if any two cones, supplemental one to the other, are cut by a concentric sphere,

The sum of their spherical bases, together with twice the sum of their lateral surfaces, is equal to the surface of the sphere.

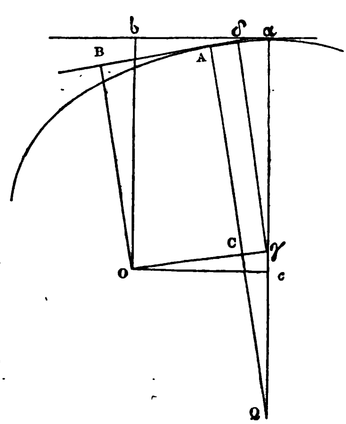
And, *The difference of their bases is equal to twice the difference of their lateral surfaces.*

Again, let a cone whose principal angles are supplemental, be cut by a concentric sphere,

The area of the spherical base, together with twice the lateral surface, is equal to the surface of the hemisphere.

XII. We shall now proceed to develope other analogies between plane and spherical curves. Having first given a simple demonstration of the well-known formula for the rectification of a plane curve $s = \int p\,d\lambda \pm h$, we shall proceed to apply this proof to establish an analogous formula for the rectification of a curve on the surface of a sphere.

Let Q be the centre of curvature of the arc of the curve at A, AB a tangent at A, OB a perpendicular from the pole O upon this tangent, OC a perpendicular upon the radius of curvature QA. Then $\text{OC} = \text{AB} = h$. Let a, b, c, O, Q be the corresponding points in the construction for a point a on the curve indefinitely near to A. Produce OC to meet Qa in γ, and through γ let $\gamma\delta$ be drawn parallel to the radius of curvature QA. Now it is plain, from the construction above given, that Aa, the element of the curve, is equal to $\text{A}\delta + \delta a$.



$\text{A}\delta = \text{C}\gamma = \text{O}c - \text{OC} = ab - \text{AB} = h' - h = dh$. We also have $\delta a = \gamma\delta \times \text{A}qa = \text{OB}. \times \text{angle } \text{A}qa = p\,d\lambda$, writing p for the perpendicular OB, and λ for the angle made by this perpendicular with a fixed axe passing through O. Introducing these ultimate ratios, writing

ds for the element of the curve, and taking $d\lambda$ as the independent variable,

$$\frac{ds}{d\lambda} = p + \frac{dh}{d\lambda}.$$

In the proof of this theorem now given, the radius of curvature is assumed to be greater than p. Should it however be less, the expression will in that case become

$$\frac{ds}{d\lambda} = p - \frac{dh}{d\lambda}.$$

Integrating this expression, we obtain the equation

$$s = \int p\,d\lambda \pm h \quad . \quad . \quad . \quad (30).$$

Should the points O and Q be on opposite sides of the curve the formula will become

$$s = h - \int p\,d\lambda.$$

XIII. It is obvious that when the perpendicular p is equal to the radius of curvature, at such a point of the curve $\frac{dh}{d\lambda} = 0$ or h is there either a maximum or a minimum.

XIV. It may also be shown that $\frac{dp}{d\lambda} = h$; for $c\gamma = ca - \gamma a = ca - \text{CA} = p' - p = dp$; we have also $oc = h + dh$ and the angle $\text{C}oc = d\lambda$, whence $dp = (h + dh)\,d\lambda$, or omitting $dh \,.\, d\lambda$ as being a quantity of the second order, we obtain the limiting value

$$h = \frac{dp}{d\lambda} \quad . \quad . \quad . \quad . \quad (31).$$

XV. We shall now proceed to establish a formula analogous to (30), for the rectification of a spherical curve, the intersection of a cone of any order with a concentric sphere.

Let a point be assumed on the surface of the sphere as pole, and through this point a tangent plane (Σ) to the sphere being drawn, the cone whose vertex is at the centre of the sphere, and which passes through the given spherical curve, will cut this

tangent plane in a plane curve where rectification may be effected, when possible, by the formula (30). Now, a tangent plane (T) may be conceived as drawn touching the cone, and cutting the tangent plane (Σ) to the sphere in a right line h, which will be a tangent to the plane curve, and also cutting the sphere in an arc of a great circle touching the spherical curve. Let the distance of the point of contact of the line h with the plane curve from the centre of the sphere be R. Through the centre of the sphere let a plane (Π) be drawn at right angles to the right line h. Now this plane (Π), as it is drawn perpendicular to h, is perpendicular also to the planes (Σ) and (T), which pass through h. As it is perpendicular to the plane (Σ), it must pass through the point of contact of (Σ) with the sphere and cut the plane curve in a right line p, which passes through the pole, the point of contact of (Σ) with the sphere. This line p being in (Π) must be perpendicular to h. The plane (Π) will also cut the surface of the sphere in an arc ϖ, of a great circle perpendicular to the tangent arc to the spherical curve; for these arcs must be perpendicular to each other, since the planes in which they lie, (Π) and (T), are at right angles, and pass through the centre of the sphere. Let P be the distance of the point in which the plane (Π) cuts the right line h, to the centre of the sphere, r the distance of the pole of the plane curve to the point in which h touches it; then, to those who follow this construction, it will be manifest (k being the radius of the sphere) that

$$\mathrm{R}^2 = k^2 + r^2, \ \mathrm{P}^2 = k^2 + p^2, \ \mathrm{R}^2 = \mathrm{P}^2 + h^2 \quad . \quad (32).$$

Let ds be the element of an arc of the plane curve between any two consecutive positions of R indefinitely near to each other; $k\,d\sigma$ the corresponding element of the spherical curve between the same consecutive positions of R. Then the areas of the elementary triangles on the surface of the cone between these consecutive positions of R, having their vertices at the centre of the sphere, and for bases the elements of the arcs of the plane and spherical curves respectively, are as their bases multiplied by their altitudes. Let D and Δ be those areas, then

$$\mathrm{D} : \Delta :: \mathrm{P}ds : k d\sigma \times k.$$

But the areas of triangles are also as the products of the sides

into the sines of the contained angles, *i. e.* as the squares of the sides into the contained elementary angles, or

$$\text{D} : \Delta :: \text{R}^2\, d\sigma : k^2\, d\sigma : \text{R}^2 : k^2,$$

eliminating the ratio D : Δ

$$d\sigma = \frac{\text{P}}{\text{R}^2}\, ds \quad . \quad . \quad . \quad (33).$$

Let λ be the angle which p or ϖ makes with the arc of a fixed great circle drawn through the pole. We may take this quantity as the independent variable. Then the last equation becomes

$$\frac{d\sigma}{d\lambda} = \frac{\text{P}}{\text{R}^2}\frac{ds}{d\lambda};$$

Substituting in the formula $\frac{ds}{d\lambda} = p + \frac{dh}{d\lambda}$ given in Sect. (XII.)

the value of $\frac{ds}{d\lambda}$ as expressed in the preceding equation, we find

$$\frac{d\sigma}{d\lambda} = \frac{\text{P}p}{\text{R}^2} + \frac{\text{P}}{\text{R}^2}\frac{dh}{d\lambda} \quad . \quad . \quad . \quad \text{(a)}:$$

Now, ϖ being the arc which p subtends at the centre of the sphere, $p = \text{P} \sin \varpi$, $\text{P}^2 = \text{R}^2 - h^2$; introducing those transformations into the last formula,

$$\frac{d\sigma}{d\lambda} = \sin \varpi + \frac{1}{\text{R}^2}\left\{ \text{P}\frac{dh}{d\lambda} - h^2 \sin \varpi \right\} . \quad \text{(b)}.$$

$$\text{But } \sin \varpi = \frac{p}{\text{P}},\ h = \frac{dp}{d\lambda},\ \frac{dh}{d\lambda} = \frac{d^2 p}{d\lambda^2},\ \text{P}\frac{d\text{P}}{d\lambda} = p\,\frac{dp}{d\lambda} \quad \text{(c)};$$

making those substitutions in the preceding equation,

$$\frac{d\sigma}{d\lambda} = \sin \varpi + \frac{1}{\text{R}^2}\left\{ \text{P}\frac{d^2p}{d\lambda^2} - \frac{d\text{P}}{d\lambda}\cdot\frac{dp}{d\lambda}\right\} \quad . \quad \text{(d)}.$$

We now proceed to show that the last term of this equation is the differential of the arc, with respect to λ, subtended by h at the centre of the sphere.

Let this arc be τ; then $\tan \tau = \frac{h}{\text{P}}$, $\cos \tau = \frac{\text{P}}{\text{R}}$

differentiating the former of those equations, and eliminating $\cos \tau$ by the help of the latter,

$$\frac{d\tau}{d\lambda} = \frac{1}{\text{R}^2}\left\{ \text{P}\frac{dh}{d\lambda} - h\frac{d\text{P}}{d\lambda}\right\} .$$

$$\text{(31) gives } h = \frac{dp}{d\lambda}, \text{ whence } \frac{dh}{d\lambda} = \frac{d^2p}{d\lambda^2};$$

substituting in the last equation those values,

$$\frac{d\tau}{d\lambda} = \frac{1}{\text{R}^2}\left\{ \text{P}\frac{d^2 p}{d\lambda^2} - \frac{d\text{P}}{d\lambda}\frac{dp}{d\lambda}\right\}.$$

Subtracting this equation from (d), we obtain the final resulting equation,

$$\frac{d\sigma}{d\lambda} = \sin\varpi + \frac{d\tau}{d\lambda}, \text{ or integrating}$$

$$\sigma = \int d\lambda\,[\sin\varpi] + \tau \quad . \quad . \quad . \quad (34);$$

a formula for the rectification of a spherical curve analogous to (30) for the rectification of a plane curve. As the latter may be expressed by a definite integral and a finite right line, so may the former be represented by a definite integral and an arc of a circle.

XVI. This formula serves a twofold purpose, for it will also enable us to give the quadrature of the supplemental figure on the surface of the sphere. Let ρ' be that radius vector of the supplemental figure on the surface of the sphere which is the prolongation of ϖ, $\rho' + \varpi = \frac{\pi}{2}$, and therefore $\sin\varpi = \cos\rho'$, λ remains the same in both curves; whence

$$\int \sin\varpi\, d\lambda = \int \cos\rho'\, d\lambda.$$

But it was shown in (6) that the expression for the area of a spherical curve is

$$\text{area} = \int (1 - \cos\rho')\, d\lambda = \lambda - \int \sin\varpi\, d\lambda \quad . \quad (35).$$

Thus the proposition established in (X.) as to the reciprocal relations between the rectification and quadrature of supplemental spherical conics of the second order, is shown to hold with respect to supplemental conics of any order described on the surface of a sphere.

XVII. To apply the formula (34) to the rectification of the spherical ellipse.

Let, as before, A and B be the semiaxes of the plane elliptic base of the cone, r the central radius vector drawn to the point

of contact of the tangent h, p the perpendicular from the centre on this tangent, h the intercept between the point of contact and the foot of this perpendicular, λ the angle between p and A. Let α, β, ρ, ϖ, τ, be the angles subtended at the centre of the sphere, whose radius is k, by the lines A, B, r, p, h; we shall consequently have $A=k\tan\alpha$, $B=k\tan\beta$, $r=k\tan\rho$, $p=k\tan\varpi$, and $h=P\tan\tau$.

Now in the plane ellipse $p^2=A^2\cos^2\lambda+B^2\sin^2\lambda$; therefore, in the spherical ellipse

$$\tan^2\varpi=\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda, \quad . \qquad (36).$$

Whence, $\sec^2\varpi=\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda$ dividing the former by the latter

$$\sin^2\varpi=\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda} \quad . \quad . \quad . \qquad (37).$$

Introducing this value of $\sin\varpi$ into the general formula for spherical rectification, the resulting equation will become

$$\sigma=\int d\lambda\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}\right]^{\frac{1}{2}}-\tau. \qquad (38).$$

The sign of τ is determined by the sign of h in (30), the formula given for the rectification of the plane curve. In the plane ellipse when λ is measured from the major axe h is negative, and therefore τ also is negative.

XVIII. To reduce this equation to the usual form of an elliptic function.

Assume $\tan\phi'=\cos\epsilon\tan\lambda$, $\quad . \quad . \quad . \quad . \quad . \qquad$ (39).

ϵ being the focal angle, and therefore $\cos\epsilon=\dfrac{\cos\alpha}{\cos\beta}$, see (13).

Introducing this relation into (38), we obtain

$$\frac{d\sigma}{d\varphi'}=\frac{\cos\alpha\cos\beta\left[\sin^2\alpha-(\sin^2\alpha-\sin^2\beta)\sin^2\varphi'\right]}{\left[\cos^2\alpha+(\sin^2\alpha-\sin^2\beta)\sin^2\varphi'\right]\sqrt{\sin^2\alpha\cos^2\varphi'+\sin^2\beta\sin^2\varphi'}}-\tau$$

as $\cos\epsilon=\dfrac{\cos\alpha}{\cos\beta}$, $\tan^2\epsilon=\dfrac{\cos^2\beta-\cos^2\alpha}{\cos^2\alpha}=\dfrac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}$.

It has already been shown in (21) that $\sin^2\eta=\dfrac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}$;

making the transformations suggested by those relations, and reducing, we obtain the following equation:

$$\left.\begin{array}{l}\sigma = \dfrac{\cos\beta}{\cos\alpha\sin\alpha}\displaystyle\int\frac{d\phi'}{\{1+\tan^2\epsilon\sin^2\phi'\}\sqrt{1-\sin^2\eta\sin^2\phi'}} \\ -\dfrac{\cos\sigma\cos\beta}{\sin\alpha}\displaystyle\int\frac{d\phi'}{\sqrt{1-\sin^2\eta\sin^2\phi'}} - \tau\end{array}\right\} \quad (40);$$

an elliptic function of the *third* order, with a positive parameter, and therefore of the circular form.

XIX. We shall now proceed to show that the elliptic functions given in the preceding formula, and in (22) have the same amplitude, or that the angles of reduction φ and φ' are identical.

In an ellipse, if ψ and λ are the angles which a central radius vector, and a perpendicular from the centre, on the tangent drawn through its extremity, make with the major axis, we know that $\tan\psi = \frac{B^2}{A^2}\tan\lambda$, A and B being the semiaxes of the curve. Hence in the spherical ellipse $\tan\psi = \dfrac{\tan^2\beta}{\tan^2\alpha}\tan\lambda$. Introducing this value of $\tan\psi$ into (8), and reducing

$$\cos^2\rho = \cos^2\alpha\cos^2\beta\left[\frac{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda}{\tan^2\alpha\cos^2\beta\cos^2\lambda + \tan^2\beta\cos^2\alpha\sin^2\lambda}\right] \quad (41).$$

Comparing this value of $\cos\rho$ with its value in (19), in which it was assumed that

$$\cos^2\rho = \frac{\sin^2\alpha\cos^2\varphi + \sin^2\beta\sin^2\varphi}{\tan^2\alpha\cos^2\varphi + \tan^2\beta\sin^2\varphi},$$

after some reductions we get $\tan\varphi = \cos\epsilon\tan\lambda$. (42).

But $\tan\varphi'$ is also equal to $\cos\epsilon\tan\lambda$, being so assumed in (39); whence

$$\varphi = \varphi' \quad . \quad . \quad . \quad (43).$$

It follows from this that the amplitudes of the elliptic functions given in (22) and (40) are the same. They represent, therefore, the *same* arc of the spherical ellipse, and may consequently be equated together; we have accordingly

$$\left.\begin{array}{l}\dfrac{\tan\beta}{\tan\alpha}\sin\beta\displaystyle\int\frac{d\phi}{\{1-e^2\sin^2\phi\}\sqrt{1-\sin^2\eta\sin^2\phi}} = \\ \dfrac{\cos\beta}{\cos\alpha\sin\alpha}\displaystyle\int\frac{d\phi}{\{1+\tan^2\epsilon\sin\phi\}\sqrt{1-\sin^2\eta\sin^2\phi}} \\ -\dfrac{\cos\alpha\cos\beta}{\sin\alpha}\displaystyle\int\frac{d\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}} - \tau.\end{array}\right] \quad (44).$$

We thus see how an elliptic function, with a *positive* parameter, may be made to depend on another with a *negative* parameter, less than 1, and greater than $\sin^2\eta$.

We are thus in a position to show that, when any elliptic function of the third order and circular form is given, whether the parameter be positive or negative, we may always obtain the elements of the spherical ellipse, of whose arc the given function is the representative.

First let the parameter be negative, then as

$$e^2 = \frac{\tan^2\alpha - \tan^2\beta}{\tan^2\alpha}, \text{ and } \sin^2\eta = \frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}, \text{ see (12) and (21),}$$

reducing

$$\tan^2\alpha = \frac{e^2 - \sin^2\eta}{\sin^2\eta\,(1 - e^2)}, \quad \tan^2\beta = \frac{e^2 - \sin^2\eta}{\sin^2\eta}, \qquad (45).$$

Writing the given elliptic function in the usual form, with the ordinary rotation, and with a negative parameter,

$$\int \frac{d\phi}{(1 - m\sin^2\phi)\sqrt{1 - c^2\sin^2\phi}},$$

where $m = e^2$ and $c = \sin\eta$, we get

$$\tan^2\alpha = \frac{m - c^2}{c^2\,(1-m)}, \quad \tan^2\beta = \frac{m - c^2}{c^2} \quad . \quad (46).$$

The coefficient of the elliptic function given in (22), $\frac{\tan\beta}{\tan\alpha}\sin\beta$, is the square root of the criterion of circularity; for

$$\frac{\tan\beta}{\tan\alpha}\sin\beta = \sqrt{(1 - m)\left(1 - \frac{c^2}{m}\right)} \quad . \quad (47).$$

It is evident from the values of $\tan^2\alpha$ and $\tan^2\beta$, given in (45) or (46), that when the parameter is negative it must be less than 1 and greater than c^2, otherwise the expression for $\tan\alpha$ would be imaginary.

XX. Let the given elliptic function have a *positive* parameter. Adopting the usual notation, it may be written

$$\int \frac{d\phi}{\{1 + n\sin^2\phi\}\sqrt{1 - c^2\sin^2\phi}}$$

In (40) it was shown that $\tan^2\varepsilon = n$, $c = \sin\eta$,

But $\tan^2\varepsilon = \frac{\sin^2\alpha - \sin^2\beta}{\cos^2\alpha}$, $\sin^2\eta = \frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}$,

Whence $\tan^2\alpha = \dfrac{\tan^2\epsilon}{\sin^2\eta}$, $\tan^2\beta = \dfrac{\tan^2\epsilon\,(1-\sin^2\eta)}{\sin^2\eta\,(1+\tan^2\epsilon)}$,

$$\text{or, } \tan^2\alpha = \frac{n}{c^2},\ \tan^2\beta = \frac{n(1-c^2)}{c^2(1+n)} \quad . \quad . \quad (48).$$

The values of $\tan\alpha$, $\tan\beta$ given in the preceding equations, show that there is no restriction on the magnitude of n, when n is positive.

XXI. To express the arc τ in terms of λ and ϕ.

It was shown in Sec. (XV.) that $\tan\tau = \dfrac{h}{\text{P}}$, whence

$$\tan^2\tau = \frac{h^2}{\text{P}^2} = \frac{h^2p^2}{\text{P}^2p^2} = \frac{(\text{A}^2-\text{B}^2)\sin^2\lambda\cos^2\lambda}{\{k^2+\text{A}^2\cos^2\lambda+\text{B}^2\sin^2\lambda\}\{\text{A}^2\cos^2\lambda+\text{B}^2\sin^2\lambda\}}$$

Introducing into this equation, the relations $k\tan\alpha = \text{A}$, $k\tan\beta = \text{B}$, $e^2 = \dfrac{\text{A}^2-\text{B}^2}{\text{A}^2}$, and $\sin^2\epsilon = \dfrac{\sin^2\alpha - \sin^2\beta}{\cos^2\beta}$, it becomes

$$\tan\tau = \frac{e^2\sin\alpha\sin\lambda\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}\,\sqrt{1-\sin^2\epsilon\sin^2\lambda}} \quad . \quad (49).$$

XXII. To express $\tan\tau$ in terms of the amplitude ϕ.

Assuming the relation established in (39), $\tan\phi = \cos\epsilon\tan\lambda$, and introducing this condition into (49) we obtain,

$$\tan\tau = \frac{e\tan\epsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}} \quad . \quad . \quad (50).$$

Substituting this value of $\tan\tau$ in (44), and transposing

$$\left.\begin{aligned}
&\frac{\cos\beta}{\cos\alpha\sin\alpha}\int\frac{d\phi}{\{1+\tan^2\epsilon\sin^2\phi\}\sqrt{1-\sin^2\eta\sin^2\phi}}\\
&-\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{\{1-e^2\sin^2\phi\}\sqrt{1-\sin^2\eta\sin^2\phi}}\\
&=\frac{\cos\alpha\cos\beta}{\sin\alpha}\int\frac{d\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}+\tan^{-1}\left\{\frac{e\tan\epsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}\right\}
\end{aligned}\right]\ (51).$$

If we now adopt the notation of Legendre, and write n for $\tan^2\epsilon$, m for e^2, c for $\sin\eta$, and thence deduce the values of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$, in terms of m, n, c, we shall meet with

the well-known formula for the comparison of the two circular forms of elliptic functions of the third order,

$$\left.\begin{array}{l}\left(\frac{1+n}{n}\right)\ \Pi_c\,(n,\varphi) - \left(\frac{1-m}{m}\right)\Pi_c\left(-m,\varphi\right) = \\ \frac{c^2}{mn}\,\mathrm{F}c\left(\varphi\right) + \frac{1}{\sqrt{mn}}\ \tan^{-1}\left\{\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\right\}\end{array}\right] \quad . \quad (52).$$

We have thus obtained the true geometrical interpretation of this cardinal formula in the comparison of elliptic functions. It follows, therefore, that the circular forms of the third order, whether the parameters be positive or negative, are the transcendental expressions for the arcs of spherical ellipses, whose principal arcs, and therefore the ellipses themselves, may be determined from the constants of the given expression by the help of (46) or (48). The extent of the arc may in like manner be defined from the equation connecting the amplitude with the angle which the perpendicular arc, from the centre on the tangent, makes with the major principal arc. This angle may for distinctness be termed the *normal angle.*

XXIII. It remains now to exhibit the class of spherical conic sections whose rectification and quadrature may be effected by elliptic functions of the *first* order. M. Gudermann in Crelle's Journal, and M. Catalan in Lionville's Journal, have shown that the rectification and quadrature of the spherical ellipse depend on elliptic functions of the third order and circular form. It has not however been shown hitherto, so far as the author is aware, that the curve whose rectification and quadrature depends on an elliptic function of the first order, is also a spherical conic section, whose principal arcs are connected by a simple relation between the tangents of the semiangles of the generating cone. Geometers, indeed, while searching for the type of this order, have been led to the consideration of certain curves more or less complex, whose rectification might be effected by an elliptic function of the first order. Of those curves, one of the simplest is that devised by Legendre — the envelope of perpendiculars drawn to the extremities of the diameter of a plane ellipse. No analogy, however, was shown to exist between the geometrical relations of the curves so constructed,

and the fundamental properties of functions of this order. Those properties have had no geometrical representative. No geometrical interpretation, for example, was given of the modular transformations of Lagrange, by which the moduli and amplitudes of a series of elliptic functions are connected together, either in an ascending or descending series, until the expressions terminate at the extreme limits in circular or parabolic arcs. Those properties we shall now proceed briefly to develope.

The curve which is the gnomonic projection of the parabola, the focus being the pole, may be rectified by an elliptic function of the first order.

Let a sphere be described touching the plane of the parabola at its focus. The spherical curve which is the intersection of the sphere with a cone whose vertex is at its centre, and whose base is the parabola, may be termed the *spherical parabola.*

To find the polar equation of this curve.

The polar equation of the parabola, the focus being the pole is $r = \frac{2g}{1 + \cos\psi}$, g being one fourth of the parameter of the parabola. Let γ be the angle which g subtends at the centre of the sphere, and ρ the angle subtended by r, then,

$$\tan\rho = \frac{2\tan\gamma}{1 + \cos\psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

Let p' be the perpendicular from the focus on a tangent to the parabola, μ the angle which this perpendicular makes with the axe, then $p' = \frac{g}{\cos\mu}$; whence in the spherical curve as $p' = k\tan\varpi'$, $g = k\tan\gamma$

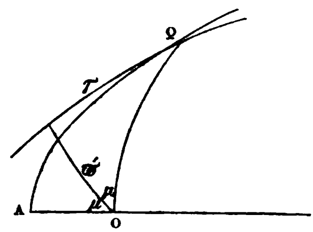
$$\tan\varpi' = \frac{\tan\gamma}{\cos\mu} \quad \text{and}$$

$$\sin\varpi' = \frac{\sin\gamma}{\sqrt{1 - \cos^2\gamma\sin^2\mu}} \quad . \quad . \quad (54).*$$

* The expression for a perpendicular arc from the focus of any spherical ellipse on a tangent arc to it may be found as follows:—

The spherical triangle, p. 13, art. iv., FOF', in which OFF' $= \mu$, OF $= \frac{\pi}{2} - \varpi'$,

Introducing this value of $\sin \varpi'$ into the general formula for spherical rectification (34), we obtain,



$$\sigma = \sin\gamma \int \frac{d\mu}{\sqrt{1-\cos^2\gamma \sin^2\mu}} + \tau.$$

Now as τ, ϖ', and μ are the sides, and an angle of a right-angled spherical triangle, since $2\mu = \psi$, we get by Napier's rules, $\tan\tau = \sin\varpi' \tan\mu$, whence

$$\sigma = \sin\gamma \int \frac{d}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\} \quad (55).$$

XXIV. When the sphere becomes indefinitely great, the spherical parabola approaches in its contour indefinitely near to the plane parabola. k being the radius of the sphere, $\sin\gamma = \tan\gamma = \frac{g}{k}$; since γ in this case is indefinitely small: whence $\cos^2\gamma = 1$, and $k\,d\sigma = ds$. In this manner (55) may be transformed into

---

$$\text{OF}' = \frac{\pi}{2} - \varpi'', \text{ gives } \cos\mu = \frac{\sin\varpi'' - \cos 2\epsilon \sin\varpi'}{\sin 2\epsilon \cos\varpi'};$$

from (5) we have $\sin\varpi' \sin\varpi'' = \sin(\alpha+\epsilon)\sin(\alpha-\epsilon)$, eliminating $\sin\varpi''$ between those equations, we obtain, after some reductions,

$$\sin^2\varpi' = \frac{2\sin^2\epsilon.\cos^2\epsilon\cos^2\mu + (\sin^2\alpha - \sin^2\epsilon)\cos 2\epsilon \pm \sin\epsilon\cos\epsilon\cos\mu\sqrt{\sin^2 2\alpha - \sin^2 2\epsilon\sin^2\mu}}{(1-\sin^2 2\epsilon\sin^2\mu)}.$$

When the curve is the spherical parabola, $\alpha + \epsilon = \frac{\pi}{2}$, $\alpha - \epsilon = \gamma$,

and the preceding expression becomes $\sin\varpi' = \dfrac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}$ or $\sin\varpi' = 1$

as we take the sign $-$ or $+$.

The locus of the foot of this perpendicular is a great circle touching the spherical parabola at its vertex. Draw the tangent circle at A, and produce the perpendicular ϖ' until it meets this tangent circle in D. Write δ for this produced perpendicular arc. Hence in the right-angled spherical triangle DOA, $\cos\mu = \tan\gamma\cot\delta$, or $\tan\delta = \frac{\tan\gamma}{\cos\mu}$. But $\tan\varpi' = \frac{\tan\gamma}{\cos\mu}$. Whence $\varpi' = \delta$. The second value of ϖ' when the circle is drawn touching the spherical parabola at the other vertex B is $\frac{\pi}{2}$, as shown above. This is manifestly the true value of ϖ', since the focus F is the pole of the great circle touching the curve at B.

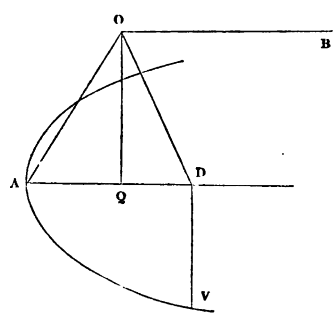
$$s = g\int\frac{d\mu}{\cos\mu} + g\frac{\sin\mu}{\cos^2\mu},$$ the established formula for the rectification of the plane parabola.

When, on the other hand, the sphere is indefinitely small, compared with the parabola, γ approximates to a right angle, and (55) becomes modified into

$s = \mu + \tan^{-1}(\tan\mu) = 2\mu$, as it should be, for 2μ is the angle which the radius vector ρ makes with the axis.

We shall presently find the notice of those extreme cases useful.

XXV. Although this curve has been named the spherical parabola, as indicating the mode of generation, it is in fact a closed curve, like all other curves which are the intersections of cones of the second degree with concentric spheres. It is a spherical ellipse, and we shall now proceed to determine its principal arcs.



Let A D V be a parabola, Q its focus, O being the centre of the sphere, which touches the plane of the parabola at Q, and being also the vertex of the obtuse-angled cone, of which the parabola A D V is a section parallel to the side of the cone O B. Let Q O A $= \gamma$, α and β being the principal semiangles of the cone,

$2\alpha = \frac{\pi}{2} + \gamma$, or $\alpha = \frac{\pi}{4} + \frac{\gamma}{2}$; whence

$$\tan^2\alpha = \left\{\frac{1+\tan\frac{\gamma}{2}}{1-\tan\frac{\gamma}{2}}\right\}^2 = \left\{\frac{\cos\frac{\gamma}{2}+\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}-\sin\frac{\gamma}{2}}\right\}^2$$

or $\tan^2\alpha = \frac{1+\sin\gamma}{1-\sin\gamma}$.

To determine the angle β.

Bisect the vertical angle A O B of the cone by the right line O D,

draw the ordinate DV of the parabola. Then $\tan^2\beta = \frac{\overline{DV}^2}{\overline{OD}^2}$. As AOD is an isosceles triangle, $AD = AO = \frac{OQ}{\cos\gamma}$. $OD = \frac{OQ}{\sin\alpha} = \frac{OQ}{\sin\left(\frac{\pi}{4} + \frac{\gamma}{2}\right)}$, and $AQ = OQ \tan\gamma$. We have also, as DV is an ordinate of the parabola, $\overline{DV}^2 = 4\,AQ.\,AD = 4\,\overline{OQ}^2\,\frac{\sin\gamma}{\cos^2\gamma}$ hence substituting

$$\tan^2\beta = \frac{2\sin\gamma}{1 - \sin\gamma}.$$

We may hence deduce the following important proposition:

The spherical ellipse, whose principal arcs are given by the equations

$$\tan^2\alpha = \frac{1+\sin\gamma}{1-\sin\gamma},\ \tan^2\beta = \frac{2\sin\gamma}{1-\sin\gamma} \quad . \quad (56)$$

(γ being any arbitrary angle), *may be rectified by an elliptic function of the first order.*

Write x for $\tan\alpha$, y for $\tan\beta$, and eliminate $\sin\gamma$ from the preceding equations, the resulting equation will become

$$x^2 - y^2 = 1,$$

the equation of an equilateral hyperbola. We thus arrive at the following proposition:

Any spherical conic section, the tangents of whose principal semiarcs are the ordinates of an equilateral hyperbola, whose transverse semiaxis is 1, *may be rectified by an elliptic function of the first order. The quadrature of a spherical conic may be effected by an elliptic function of the first order, when the cotangents of the halves of the principal semiangles of the cone are the ordinates of an equilateral hyperbola, whose transverse semiaxis is* 1.

XXVI. When we take the complete function (55), or in-

tegrate between the limits 0 and $\frac{\pi}{2}$; we get, not the length of a quadrant of the spherical parabola, as we do when we take the centre as origin, but the length of two quadrants or half the ellipse. We derive also this other remarkable result, that when μ is a right angle, the spherical triangle, whose sides are the radius vector, the perpendicular arc on the tangent, and the intercept of the tangent arc between the point of contact and the foot of the perpendicular, is a quadrantal right-angled triangle, or its sides are all quadrants. For when $\mu = \frac{\pi}{2}$

$$\rho = \frac{\pi}{2}, \ \varpi' = \frac{\pi}{2}, \ \tau = \frac{\pi}{2}.$$

XXVII. It may easily be shown that the arc of a great circle, which touches the spherical parabola, intercepted between the perpendicular arcs let fall upon it from the foci, is in every position constant and equal to a quadrant.*

If we take the spherical conic supplemental to the given parabola, the foci of this latter are the extremities of the minor principal arc of the former, and the cyclic arcs of the former are tangents to the latter at the extremities of its major principal arc.

XXVIII. Resuming the equations given in (56), which express the tangents of the principal semiarcs of the spherical parabola in terms of sin γ, namely,

$$\tan^2\alpha = \frac{1+\sin\gamma}{1-\sin\gamma}, \ \tan^2\beta = \frac{2\sin\gamma}{1-\sin\gamma},$$

we may easily deduce

$$\tan^2\varepsilon = \frac{1-\sin\gamma}{1+\sin\gamma}, \ e^2 = \frac{1-\sin\gamma}{1+\sin\gamma}, \ \sin^2\eta = \left\{\frac{1-\sin\gamma}{1+\sin\gamma}\right\}^2 \qquad (57)$$

whence $\tan^2\varepsilon = e^2 = \sin\eta = \cos^2\beta$.

* As $\sin\varpi' = \frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}$, and $\sin\varpi'\sin\varpi'' = \sin\gamma$, see (5), we must have $\sin\varpi'' = \sqrt{1-\cos^2\gamma\sin^2\mu}$. Hence as $\varpi' = \frac{\pi}{2} - \text{FO}$, $\varpi'' = \frac{\pi}{2} - \text{F}'\text{O}$, $\cos \text{FO} . \cos \text{FO}' = \sin\gamma = \cos \text{FF}$: or the angle FOF′ is a right angle. (*Fig.* p. 13.)

XXIX. We now proceed to rectify the spherical parabola by the help of the ordinary formulæ for rectification, the centre being the pole: by this method some very curious geometrical results will be obtained, which have hitherto appeared as mere analytical expressions. In (22), the following expression was established for the arc of a spherical ellipse measured from the major arc, the centre being the pole.

$$\sigma = \frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{(1-e^2\sin^2\phi)\sqrt{1-\sin^2\eta\sin^2\phi}}$$

Or substituting the values of the constants given in the preceding equations.

$$\sigma = \frac{2\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\left\{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin^2\phi\right\}\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} \quad (58).$$

But when the focus is the pole, we found for the arc of a spherical parabola the expression

$$\sigma = \sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\};$$

Equating those values of σ, we obtain the resulting equation

$$\left.\begin{aligned}&\frac{2\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\left\{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin^2\phi\right\}\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\\ &=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\}\end{aligned}\right] (59).$$

XXX. We shall now show that the amplitudes ϕ and μ in the preceding formula are connected by the equation

$$\tan(\phi-\mu) = \sin\gamma\tan\mu \quad . \quad . \quad . \quad (60);$$

a relation long ago established by Lagrange.

Let ϖ and ϖ' be the perpendicular arcs from the centre and focus of the spherical parabola on the tangent arc: λ and μ the angles which those perpendicular arcs make with the major principal arc. The distance between the centre and focus of the spherical parabola, with the complements of those perpendiculars, constitute the sides of a spherical triangle. We shall therefore have $\sin^2\lambda = \sin^2\mu\,\dfrac{\sec^2\varpi}{\sec^2\varpi'}$ . . . . (a).

Now $\sec^2 \varpi = \sec^2 \alpha \cos^2 \lambda + \sec^2 \beta \sin^2 \lambda$. See (36). Or, writing for $\sec \alpha$, $\sec \beta$, their particular values in the spherical parabola as given in (56),

$$\sec^2 \varpi = \frac{2-(1-\sin \gamma) \sin^2 \lambda}{1-\sin \gamma}$$

Again, as $\tan \varpi' = \frac{\tan \gamma}{\cos \mu}$, $\sec^2 \varpi' = \frac{\tan^2 \gamma + \cos^2 \mu}{\cos^2 \mu}$.

Introducing those relations into (a), we obtain, after some reductions,

$$\sin^2 \lambda = \frac{2(1+\sin \gamma) \sin^2 \mu \cos^2 \mu}{1-\cos^2 \gamma \sin^4 \mu}; \text{ or reducing}$$

$$\sin^2 \lambda = \frac{2(1+\sin \gamma)}{2+\cot^2 \mu + \sin^2 \gamma \tan^2 \mu}.$$

Adding to and subtracting from the denominator of this expression the quantity $2 \sin \gamma$, we find

$$\sin^2 \lambda = \frac{2(1+\sin \gamma)}{2(1+\sin \gamma)+(\cot \mu - \sin \gamma \tan \mu)^2}.$$

Reducing, we obtain

$$\tan^2 \lambda = \frac{2(1+\sin \gamma)}{(\cot \mu - \sin \gamma \tan \mu)^2}.$$

In the case of the spherical parabola, $\cos^2 \varepsilon = \frac{1+\sin \gamma}{2}$,

whence $\cos \varepsilon \tan \lambda = \dfrac{1+\sin \gamma}{\cot \mu - \sin \gamma \tan \mu}$ . . . (b)

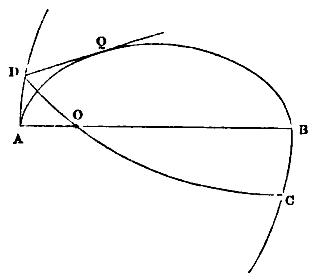
or, $\cos \varepsilon \tan \lambda = \dfrac{\tan \mu + \sin \gamma \tan \mu}{1-\tan \mu \sin \gamma \tan \mu}$.

The second member of this equation is manifestly the expression for the tangent of the sum of two arcs μ and ν, if we make $\tan \nu = \sin \gamma \tan \mu$,

hence $\cos \varepsilon \tan \lambda = \tan (\mu + \nu)$.

But in (39) and (42) it was shown that $\tan \phi = \cos \varepsilon \tan \lambda$,

whence $\tan \phi = \tan (\mu + \nu)$, or $\phi = \mu + \nu$, or $(\phi - \mu) = \nu$; whence $\tan (\phi - \mu) = \tan \nu = \sin \gamma \tan \mu$.



A geometrical interpretation of Lagrange's theorem $\tan (\phi - \mu) = \sin \gamma \tan \mu$, may easily be given by the help of the spherical parabola. At the extremities A and B of the major principal arc 2α of the spherical parabola, let great circles AD, BC, be drawn touching the curve. Then the focus O is the pole of the great circle BC. Let OD be the perpendicular arc from the focus O on the tangent QD. The point D is in the great circle touching the curve at A. (See p. 32.) Let $AD = \nu$, $AO = \gamma$, and the angle $AOD = BC = \mu$. In the right-angled spherical triangle AOD, $\sin \gamma = \tan \nu \cot \mu$, or $\tan \nu = \sin \gamma \tan \mu$. Whence $\tan (\phi - \mu) = \tan \nu$, or $\phi - \mu = \nu$,

$$\text{or } \phi = \mu + \nu. \quad \text{Whence } \phi = AD + BC;$$

or the amplitude ϕ is the sum of the arcs of two great circles, touching the spherical parabola at the extremities of the principal major arc of the curve, intercepted between those points of contact and the perpendicular arc DOC let fall from the focus O on the tangent arc DQ to the curve.

Hence while the original amplitude μ is equal to an arc of the tangent circle at B, the derived amplitude ϕ is equal to the sum of two arcs of the tangent circles drawn at A and B, and given by the same construction.

XXXI. To show that

$$\int \frac{d\mu}{\sqrt{1 - \cos^2 \gamma \sin^2 \mu}} = \frac{1}{1 + \sin \gamma} \int \frac{d\phi}{\sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}}$$

the amplitudes ϕ and μ being connected by the relation established in the last section,

$$\tan(\phi - \mu) = \sin\gamma \tan\mu.$$

From equation (b) in the last section, since $\tan\phi = \cos\epsilon \tan\lambda$, we find

$$\cot\phi = \frac{\cot\mu - \sin\gamma\tan\mu}{1+\sin\gamma} \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Differentiating this expression with respect to ϕ and μ,

$$\frac{(1+\sin\gamma)}{\sin^2\phi}\,\frac{d\phi}{d\mu} = \frac{\cos^2\mu + \sin\gamma\sin^2\mu}{\cos^2\mu\sin^2\mu} \quad . \quad \text{(d)}$$

equation (b) in the last section gives

$$\tan^2\phi = \frac{(1+\sin\gamma)^2\sin^2\mu\cos^2\mu}{(\cos^2\mu - \sin\gamma\sin^2\mu)^2} \quad . \quad . \quad \text{(g)}$$

whence, after some reductions, we obtain

$$\sin^2\phi = \frac{(1+\sin\gamma)^2\sin^2\mu\cos^2\mu}{1-\cos^2\gamma\sin^2\mu} \quad . \quad . \quad \text{(e)}$$

multiplying this formula by the modulus $\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2$, and reducing, we find

$$\frac{1}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} = \frac{\sqrt{1-\cos^2\gamma\sin^2\mu}}{\cos^2\mu + \sin\gamma\sin^2\mu}. \quad \text{(f)}$$

Multiplying together the left-hand members of the equations (d), (e), (f), and also the right-hand members together, we find, dividing by $(1+\sin\gamma)$ and integrating,

$$\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} = (1+\sin\gamma)\int\frac{d\mu}{\sqrt{-\cos^2\gamma\sin^2\mu}} \text{(61).}$$

This is the well-known relation between two elliptic integrals of the first order whose moduli are $\cos\gamma$ and $\frac{1-\sin\gamma}{1+\sin\gamma}$; or, using the common notation, whose moduli are c and $\frac{1-b}{1+b}$ respectively, b being $= \sqrt{1-c^2}$; the amplitudes being connected by the

relation $\tan(\phi - \mu) = b \tan \mu$, and $F_c(\mu) = \frac{1}{1+b} F_{c'}(\phi)$.

XXXII. When a quadrant of the spherical parabola is taken, $\phi = \frac{\pi}{2}$, and (60) gives $\tan \mu = \frac{1}{\sqrt{\sin \gamma}}$, and $\tau = \frac{\pi}{4}$.

Let τ be the arc whose tangent is $\frac{\sin \gamma \tan \mu}{\sqrt{1 - \cos^2 \gamma \sin^2 \mu}}$,

$$\text{then } \tan 2\tau = \frac{2 \sin \gamma \sin \mu \cos \mu \sqrt{1 - \cos^2 \gamma \sin^2 \mu}}{\cos^4 \mu - \sin^2 \gamma \sin^4 \mu}, \quad . \quad (j)$$

and if we multiply together (f) and (g) we shall find,

$$\frac{\tan \phi}{\sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}} = \frac{(1 + \sin \gamma) \sin \mu \cos \mu \sqrt{1 - \cos^2 \gamma \sin^2 \mu}}{\cos^4 \mu - \sin^2 \gamma \sin^4 \mu} \quad (k)$$

comparing together (j) and (k), we get,

$$\tan 2\tau = \frac{\frac{2 \sin \gamma}{1 + \sin \gamma} \tan \phi}{\sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}}; \text{ or}$$

$$\tau = \tfrac{1}{2} \tan^{-1} \left\{ \frac{\left(\frac{2 \sin \gamma}{1 - \sin \gamma}\right) \tan \phi}{\sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}} \right\} \quad (62).$$

We are thus enabled to express τ the portion of the tangent arc between the point of contact and the foot of the perpendicular on it from the focus in terms of ϕ instead of μ.

If now we introduce this value of τ into (59), and combine with it the relations established in (61), the resulting equation will become

$$2 \int \frac{d\phi}{\left\{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right) \sin^2 \phi\right\} \sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}} = \left[\int \frac{d\phi}{\sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}} + \left(\frac{1 + \sin \gamma}{2 \sin \gamma}\right) \tan^{-1} \left[\frac{\left(\frac{2 \sin \gamma}{1 + \sin \gamma}\right) \tan \phi}{\sqrt{1 - \left(\frac{1 - \sin \gamma}{1 + \sin \gamma}\right)^2 \sin^2 \phi}} \right] \right] \quad (63).$$

Adopting the ordinary notation of elliptic functions

$$n = -c = \frac{1-\sin\gamma}{1+\sin\gamma}, \text{ whence } 1 + c = \frac{2\sin\gamma}{1+\sin\gamma}$$

the last formula will become

$$2\,\Pi_c(-c, \phi) = F_c(\phi) + \frac{1}{1+c}\tan^{-1}\left\{\frac{(1+c)\tan\phi}{\sqrt{1-c^2\sin^2\phi}}\right\}$$

In the *Traité des Fonctions Elliptiques*, tom. i. page 68., we meet with the formula,

$$\Pi_c(n, \phi) + \Pi_c\left(\frac{c^2}{n}, \phi\right) = F_c(\phi) + \frac{1}{\sqrt{\alpha}}\tan^{-1}\left\{\frac{\alpha\tan\phi}{\sqrt{1-c^2\sin^2\phi}}\right\} \quad (64).$$

Now when $n = -c$, this formula becomes

$$2\,\Pi_c(-c, \phi) = F_c(\phi) + \frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\phi}{\sqrt{1-c^2\sin^2\phi}}\right] \quad (65).$$

Whence (63) and (65) are identical.

XXXIII. Let us now proceed to rectify the spherical parabola by the formula for rectification given in (40), the centre being the pole. For this purpose resuming the formula for spherical rectification established in (40), and deducing the values of the parameter, modulus, and coefficients in that formula from the given relations,

$$\tan^2\alpha = \frac{1+\sin\gamma}{1-\sin\gamma}, \quad \tan^2\beta = \frac{2\sin\gamma}{1-\sin\gamma};$$

we get the parameter $\tan^2\varepsilon = \cos^2\beta = \dfrac{1-\sin\gamma}{1+\sin\gamma}$ . . (a)

The modulus $\sin\eta = \cos^2\beta = \left(\dfrac{1-\sin\gamma}{1+\sin\gamma}\right)$ . . (b).

The coefficient $\dfrac{\cos\beta}{\sin\alpha\cos\alpha} = \dfrac{2}{1+\sin\gamma}$. (c).

The coefficient $\dfrac{\cos\alpha\cos\beta}{\sin\alpha} = \dfrac{1-\sin\gamma}{1+\sin\gamma}$. (d).

$$e\tan\varepsilon = \frac{1-\sin\gamma}{1+\sin\gamma} \quad . \quad . \quad (f).$$

making those substitutions in (40), the resulting equation becomes

$$\text{arc} = \frac{2}{1+\sin\gamma}\int\frac{d\phi}{\left\{1+\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin^2\phi\right\}\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}$$

$$-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} - \tan^{-1}\left[\frac{\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin\phi\cos\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\right]$$

But from (55) the focus being the pole, we get

$$\text{arc} = \sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right]$$

In (61) it was shown that

$$\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} = \frac{1}{1+\sin\gamma}\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}$$

Introducing this relation into the last formula, and equating together the equivalent expressions for the arcs, we obtain the resulting equation:

$$\left.\begin{aligned}&2\int\frac{d\phi}{\left\{1+\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin^2\phi\right\}\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} = \int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\\ &+(1+\sin\gamma)\tan^{-1}\left\{\frac{\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin\phi\cos\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\right\} + (1+\sin\gamma)\tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\}\end{aligned}\right\} (67).$$

XXXIV. We shall now proceed to show that the common formula for the comparison of elliptic transcendents with the same modulus and amplitude, but reciprocal parameters, is identical with the geometrical theorem just established. To prove this, it must be shown that $\frac{1}{(1+n)}\tan^{-1}\left\{\frac{(1+n)\tan\phi}{\sqrt{1-n^2\sin^2\phi}}\right\}$

$$=(1+\sin\gamma)\tan^{-1}\left\{\frac{e\tan\epsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}\right\}+(1+\sin\gamma)\tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\}.$$

If we write τ, θ, ϑ, for those angles respectively, and $\tan^2\epsilon$ for n,

or $n = \frac{1-\sin\gamma}{1+\sin\gamma}$, we have to show that

$$\tau = 2\,(\theta + \vartheta)$$

$(\theta + \vartheta)$ is the arc of the great circle, which touches the spherical parabola, intercepted between the perpendicular arcs, let fall from the focus and centre upon it.

We must, in the first place, by the help of the equation of Lagrange between the amplitudes, established on geometrical principles in (xxx.), reduce those angles to a single variable. μ is taken as the independent variable instead of ϕ, as the trigonometrical function of μ in terms of ϕ is in the first power only. We have therefore,

$$\left.\begin{aligned}
\tan\tau &= \frac{2\tan\phi}{(1+\sin\gamma)\sqrt{1-\sin^2\eta\sin^2\phi}} \\
\tan\theta &= \frac{\dfrac{(1-\sin\gamma)}{(1+\sin\gamma)}\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}} \\
\tan\vartheta &= \frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}
\end{aligned}\right\} \quad . \quad . \quad (68).$$

The relation between the amplitudes ϕ and μ is given by the equation of Lagrange;

$$\tan(\phi-\mu) = \sin\gamma\tan\mu, \text{ or } \tan\phi = \frac{(1+\sin\gamma)\sin\mu\cos\mu}{\cos^2\mu - \sin\gamma\sin^2\mu}$$

Eliminating ϕ by the help of this expression from the value of $\tan\theta$ given in the preceding group, we find

$$\tan\theta = \frac{(1-\sin\gamma)\sin\mu\cos\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} \times \frac{\cos^2\mu + \sin\gamma\sin^2\mu}{\cos^2\mu - \sin\gamma\sin^2\mu}$$

using this transformation and reducing

$$\tan(\theta+\vartheta) = \tan\mu\sqrt{1-\cos^2\gamma\sin^2\mu} \quad . \quad . \quad (69);$$

a very elegant expression for the length of the tangent arc to the spherical parabola between the perpendicular arcs let fall from the centre and focus upon it.

From the last equation may be obtained

$$\tan 2(\theta+\vartheta) = \frac{2\sin\mu\cos\mu\sqrt{1-\cos^2\gamma\sin^2\mu}}{\cos^4\mu - \sin^2\gamma\sin^4\mu} \quad . \quad (70).$$

Using the preceding transformations, it may also be shown that

$$\tan \tau = \frac{2 \sin \mu \cos \mu \sqrt{1 - \cos^2 \gamma \sin^2 \mu}}{\cos^4 \mu - \sin^2 \gamma \sin^4 \mu}$$

whence $\tan \tau = \tan 2 (\theta + \vartheta)$, or

$$\tau = 2 (\theta + \vartheta) \quad . \quad . \quad . \quad . \quad (71).$$

We have thus shown that in the particular case of the general formula for comparing elliptic functions of the third order with reciprocal parameters, when *the parameter* is *positive* and equal to the *modulus*, the circular arc in the formula of comparison (67) is twice the portion of the arc of the great circle, touching the curve, and intercepted between the perpendicular arcs, let fall from the focus and centre upon it.

If we take the parameter with a *negative* sign, the circular arc in the expression (63) will represent twice the tangent arc between the point of contact and the focal perpendicular. We are thus in a position to give the geometrical meanings of the angular function in Legendre's theorem (64), whether the paramater be positive or negative.

XXXV. The foregoing investigations will have furnished us with the geometrical interpretation of the transformation of Lagrange. The preceding steps will sufficiently indicate the rationale of the following construction.

Let c be the modulus of the given elliptic integral. We may conceive a plane parabola at its focus in contact with a sphere; such that one fourth its parameter shall subtend at the centre an angle whose cosine is c. The central projection of this parabola, on the sphere will be a spherical parabola. At the centre of this spherical parabola let us imagine another plane parabola in contact at its focus with the sphere, and having its axis in the same plane with the former, of such magnitude, that one fourth of its parameter shall subtend at the centre of the sphere an angle whose cosine $= \dfrac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}} = \dfrac{1 - b}{1 + b}$

let this cosine be c'. We may repeat this construction succes-

sively until the parameter of the last of the applied tangent plane parabolas shall become so indefinitely small, compared with the radius of the sphere, that it may ultimately be taken to coincide with its projection. We shall in this way reduce, geometrically at least, the calculation of an elliptic integral of the first order to the rectification of an arc of a plane parabola, that is to a logarithm as was shown in (XXIV). If, on the contrary, the process be reversed, we shall at length arrive at a parabola whose parameter will be so large, compared with the radius of the sphere, that the central projection of this parabola will become a great circle of the sphere. The evaluation of the elliptic integral will therefore ultimately be reduced to the rectification of a circular arc. These are the well-known results of the modular transformations of Lagrange.

Let 2ε, $2\varepsilon'$, $2\varepsilon''$, $2\varepsilon'''$, &c. denote the distances between the foci of the successive spherical parabolas generated in the manner above mentioned, then it may easily be shown, writing c for $\cos\gamma$ and b for $\sin\gamma$, that

$$\sin 2\varepsilon = c,\ \sin 2\varepsilon' = \frac{1-b}{1+b},\ \sin 2\varepsilon'' = \left[\frac{1-b^{\frac{1}{2}}}{1+b^{\frac{1}{2}}}\right]^2$$

$$\sin 2\varepsilon''' = \left[\frac{(1+b)^{\frac{1}{2}} - 2^{\frac{1}{2}} b^{\frac{1}{4}}}{(1+b)^{\frac{1}{2}} + 2^{\frac{1}{2}} b^{\frac{1}{4}}}\right]^2 \quad \sin 2\varepsilon'''' = \left[\frac{1+b^{\frac{1}{2}} - 2^{\frac{1}{2}}2^{\frac{1}{4}}(1+b)^{\frac{1}{4}}b^{\frac{1}{8}}}{1+b^{\frac{1}{2}} + 2^{\frac{1}{2}}2^{\frac{1}{4}}(1+b)^{\frac{1}{4}}b^{\frac{1}{8}}}\right]^2$$

$$\sin 2\varepsilon^{\mathrm{v}} = \left[\frac{1+b+2^{\frac{1}{2}}b^{\frac{1}{4}} - 2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+b^{\frac{1}{2}})(1+b)^{\frac{1}{8}}b^{\frac{1}{16}}}{1+b+2^{\frac{1}{2}}b^{\frac{1}{4}} + 2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+b^{\frac{1}{2}})(1+b)^{\frac{1}{8}}b^{\frac{1}{16}}}\right]^2 \ \&c.$$

We perceive that the successive values of these expressions approximate continually to 1, or $2\varepsilon'''\cdots$ to $\frac{\pi}{2}$, and as generally

$$2\varepsilon'''\cdots = \frac{\pi}{2} - \gamma'''\cdots,\ \gamma''' \text{ approximates continually to } 0.$$

XXXVI. When the parameter of the elliptic integral of the third order is negative, and at the same time greater than 1, or less than the square of the modulus, the function no longer represents any spherical curve of the second order. It is pos-

sible, however, to construct a spherical curve whose rectification may be effected by an elliptic function of the third order, and logarithmic form.

Let us conceive a spherical curve which shall cut all its spherical radii-vectores in angles whose cosines shall have a given ratio to the sines of double the angles which the equal radii-vectores of a certain spherical ellipse make with the major arc. Let i be this angle, and ρ the distance of the point from the centre of the curve. In the spherical ellipse, of which the principal arcs are α and β, let this radius vector ρ make with the major arc the angle ψ. Then by the law of the generation of the curve,

$$\cos i = m \sin\psi \cos\psi \quad . \quad . \quad . \quad . \qquad (72).$$

Now as the radii of the ellipse which are equal to α and β respectively, make with the major arc angles 0 and $\frac{\pi}{2}$; at these distances $\cos i = 0$, and the curve has therefore apsides at those distances from the centre.

XXXVII. To find the length of this curve.

As $\cos i = m \sin\psi \cos\psi$, (this relation may be taken as the definition of the curve) and $\cos i = \frac{d\rho}{d\sigma}$, $m^2 \frac{d\sigma^2}{d\rho^2} = \frac{1}{\sin^2\psi \cos^2\psi}$ (a).

The equation of the spherical ellipse gives,

$$\cot^2\rho = \cot^2\alpha \cos^2\psi + \cot^2\beta \sin^2\psi$$

Let ϕ be the eccentric anomaly, then $\tan\psi = \frac{\tan\beta}{\tan\alpha}\tan\phi$

Whence $\sin^2\psi = \frac{\tan^2\beta \sin^2\phi}{\tan^2\alpha \cos^2\phi + \tan^2\beta \sin^2\phi}$, $\cos^2\psi = \frac{\tan^2\alpha \cos^2\phi}{\tan^2\alpha \cos^2\phi + \tan^2\beta \sin\phi}$ (b).

Substituting those values of $\sin\psi$, $\cos\psi$ in (a), we find,

$$\frac{m^2\, d\sigma^2}{d\rho^2} = \frac{\{\tan^2\alpha \cos^2\phi + \tan^2\beta \sin^2\phi\}^2}{\tan^2\alpha \tan^2\beta \sin^2\phi \cos^2\phi} \quad . \quad . \qquad (c).$$

Again, as $\tan^2\rho = \tan^2\alpha\cos^2\phi + \tan^2\beta\sin^2\phi$. (d)

differentiating $\dfrac{d\rho^2}{d\phi^2} = \dfrac{(\tan^2\alpha - \tan^2\beta)^2\sin^2\phi\cos^2\phi}{\tan^2\rho\sec^4\rho}$

whence as $m\dfrac{d\sigma}{d\phi} = m\dfrac{d\sigma}{d\rho}\cdot\dfrac{d\rho}{d\phi}$

$$\frac{m^2 d\sigma^2}{d\phi^2} = \frac{(\tan^2\alpha - \tan^2\beta)^2}{\tan^2\alpha\tan^2\beta}\,\frac{[\tan^2\alpha\cos^2\phi + \tan^2\beta\sin^2\phi]}{[\sec^2\alpha\cos^2\phi + \sec^2\beta\sin^2\phi]^2}$$

Now $\dfrac{\tan^2\alpha - \tan^2\beta}{\tan^2\alpha} = e^2, \dfrac{\sec^2\alpha - \sec^2\beta}{\sec^2\alpha} = \sin^2\varepsilon$;

making those substitutions, reducing and taking the square root, the transformed equation becomes

$$m\sigma = \frac{e^2}{\tan\beta}\int\frac{d\phi}{\sqrt{1 - e^2\sin^2\phi}} - \frac{e^2\cos^2\alpha}{\tan\beta}\int\frac{d\phi}{\{1 - \sin^2\epsilon\sin^2\phi\}\sqrt{1 - e^2\sin^2\phi}} \quad (73).$$

As $e^2 > \sin^2\varepsilon$, this is an elliptic function of the third order and logarithmic form.

The condition that $m\sin\psi\cos\psi = \cos i$, must necessarily be less than 1, imposes a limiting value on the angle ψ.*

* Although not bearing directly on this subject, the following theorems may be thought worthy of notice.

Let two spherical conic sections, whose principal arcs are α and β, α' and β', be so related that

$$\tan\alpha\tan\alpha'\sin\eta = 1, \quad \sec\beta\sec\beta'\sin\eta = 1.$$

Then it will always be possible to express the sum of two arcs of those spherical ellipses by means of an arc of a spherical parabola and an arc of a circle.

Those spherical ellipses, "conjugate spherical ellipses" we may name them, have certain common properties.

They have the same cyclic circles.

Their projections on the plane of the *external* axes of the cone give concentric similar and similarly posited plane ellipses.

The magnitudes of the normal angles λ and λ', which determine the extent of the compared arcs, are given by the simple relation

$\dfrac{\tan\lambda}{\tan\lambda'} = \dfrac{1}{i}$. i being the ratio of the axes of the plane ellipses.

SECTION II.

XXXVIII. We shall now proceed to the discussion of the rotation of a rigid body round a fixed point, which is more immediately the subject of the present work. Let this point be taken as the origin of three rectangular coordinates, their direction being arbitrary as well with respect to the body as to absolute space. Let us, moreover, make the supposition that the body is not subject to the action of accelerating forces, but in a state of motion originated by a single impulse, or by any number of single impulses, which may be combined into one. This may be considered as the normal state of the rotation of a body; because if it should besides be subjected to accelerating forces, such new forces will introduce variations into the arbitrary constants of the problem. It has, moreover, the advantage of admitting a complete solution. We are not compelled to have recourse to approximations. It will be shown in the following pages, that the curves, which the final integrals represent, are spherical conic sections; curves which may as easily be determined, from the principles laid down in the last section, by means of the constants which enter into the integrals, and the amplitudes of those functions, as the arc of a circle may be ascertained, when we know its radius and the angle which the arc subtends at the centre. Hitherto there has not been any attempt made, at least so far as the author is aware, to carry the solution further than to show, that as the final integrals involve the square roots of quadrinomial expressions with respect to the independent variable, they might be reduced to the usual forms of elliptic functions. But these integrals have not been interpreted so as to give a graphic representation of the motion, by means of the properties of those functions.

Assuming the usual definition of the moment of inertia of a body with respect to a certain right line; that it is the sum of all the constituent elements of the body, each multiplied into the square of its distance from this axe, we shall briefly give the usual method of finding it.

Let the given axe make the angles λ, μ, ν, with the axes of coordinates; R being the distance of one of the elements dm

from the origin, and θ the angle which this line makes with the axe. The distance, therefore, of the particle dm from the axe is $\text{R} \sin\theta$, and the moment of inertia round this axe is the sum or integral of all the terms, such as $\text{R}^2 \sin^2\theta\, d\text{m}$, which the body affords. Writing H for the moment of inertia round this axe:

$$\text{H} = \int d\text{m}\,[\text{R}\sin\theta]^2 \quad . \quad . \quad (74).$$

The integral being extended to the whole mass of the body.

To transform this integral into another, which shall contain the rectangular coordinates xyz of the particle dm. We have

$\cos\theta = \frac{x}{\text{R}}\cos\lambda + \frac{y}{\text{R}}\cos\mu + \frac{z}{\text{R}}\cos\nu$, deriving the value of $\sin\theta$ from this expression, and substituting it in (74), we get

$$\left.\begin{array}{l} \text{H} = \cos^2\lambda \int d\text{m}\,(y^2+z^2) + \cos^2\mu \int d\text{m}\,(x^2+z^2) + \cos^2\nu \int d\text{m}\,(x^2+y^2) \\ -2\cos\mu\cos\nu \int d\text{m}\,yz - 2\cos\lambda\cos\nu \int d\text{m}\,xz - 2\cos\lambda\cos\mu \int d\text{m}\,xy \end{array}\right\} (75).$$

Now these six integrals depend solely on the assumed position of the coordinate planes with respect to the body, and not on the position of the axes, which is determined by the angles λ, μ, ν. These integrals referred to the same system of coordinates will, therefore, be the same for every assumed axe. Let them be computed and designated as follows:

$$\left.\begin{array}{l} \int d\text{m}\,(y^2+z^2) = \text{L}, \int d\text{m}\,(x^2+z^2) = \text{M}, \int d\text{m}\,(x^2+y^2) = \text{N}, \\ \int d\text{m}\,yz = \text{U}, \int d\text{m}\,xz = \text{V}, \int d\text{m}\,xy = \text{W}, \end{array}\right\} (76).$$

The value of H may now be written,

$$\text{H} = \text{L}\cos^2\lambda + \text{M}\cos^2\mu + \text{N}\cos^2\nu - 2\,\text{U}\cos\mu\cos\nu - 2\,\text{V}\cos\lambda\cos\nu - 2\,\text{W}\cos\lambda\cos\mu. \quad (77).$$

We may reduce this expression to represent a line drawn from the origin to some curved surface, by the following transformations:

let $\text{H} = \text{n}\text{P}^2$, $\text{L} = \text{n}\text{A}$, $\text{M} = \text{n}\text{B}$, $\text{N} = \text{n}\text{C}$, $\text{U} = \text{n}\text{D}$, $\text{V} = \text{n}\text{E}$, $\text{W} = \text{n}\text{F}$.

Substitute and divide by the cubical quantity n, the equation becomes

$$\mathrm{P}^2 = \mathrm{A}\cos^2\lambda + \mathrm{B}\cos^2\mu + \mathrm{C}\cos^2\nu - 2\,\mathrm{D}\cos\mu\cos\nu - 2\,\mathrm{E}\cos\lambda\cos\nu - 2\,\mathrm{F}\cos\lambda\cos\mu. \quad (78).$$

Now this, as may easily be shown, is the expression for the length of a perpendicular let fall from the centre of a surface of the second order on a tangent plane to this surface. As the coefficients L, M, N, are necessarily finite and positive, the coefficients of the surface A, B, C, which have a given ratio to the former, must also be finite and positive. The surface is therefore an ellipsoid. That the above expression represents such a perpendicular may be shown as follows.

XXXIX. The tangential equation of a surface of the second order, the origin being at the centre, is

$$1 = \mathrm{A}_{,}\,\xi^2 + \mathrm{B}_{,}\,\upsilon^2 + \mathrm{C}_{,}\,\zeta^2 + 2\,\mathrm{D}_{,}\,\zeta\upsilon + 2\,\mathrm{E}_{,}\,\xi\zeta + 2\,\mathrm{F}_{,}\,\xi\upsilon. \quad (79).$$

In this equation ξ, υ, ζ denote the reciprocals of the portions of the axes of coordinates between the origin and the variable tangent plane, supposed to envelope the surface in every successive possible position. The squared reciprocal of the perpendicular from the centre on the tangent plane is $\xi^2 + \upsilon^2 + \zeta^2$. If λ, μ, ν denote the angles which this perpendicular $\mathrm{P}_{,}$ makes with the axes of coordinates, $\cos\lambda = \mathrm{P}_{,}\,\xi$, $\cos\mu = \mathrm{P}_{,}\,\upsilon$, $\cos\nu = \mathrm{P}_{,}\,\zeta$. Substituting these values of ξ, υ, ζ, in the preceding equation, and multiplying by $\mathrm{P}_{,}^2$ we find:

$$\mathrm{P}^2_{,} = \mathrm{A}_{,}\cos^2\lambda + \mathrm{B}_{,}\cos^2\mu + \mathrm{C}_{,}\cos^2\nu + 2\,\mathrm{D}_{,}\cos\mu\cos\nu + 2\,\mathrm{E}_{,}\cos\lambda\cos\nu + 2\,\mathrm{F}_{,}\cos\lambda\cos\mu \, (80).$$

An equation which coincides with (78), if we omit the traits, as we manifestly may do: hence $\mathrm{P}_{,} = \mathrm{P}$.

If we divide (77) by P^2, and introduce the quantities ξ, υ, ζ, by the help of the equations $\cos\lambda = \mathrm{P}\,\xi$, $\cos\mu = \mathrm{P}\,\upsilon$, $\cos\nu = \mathrm{P}\,\zeta$, we get

$$\mathrm{n} = \mathrm{L}\,\xi^2 + \mathrm{M}\,\upsilon^2 + \mathrm{N}\,\zeta^2 - 2\,\mathrm{U}\,\zeta\upsilon - 2\,\mathrm{V}\,\xi\zeta - 2\,\mathrm{W}\,\xi\upsilon. \quad (81).$$

It is shown in the treatise referred to*, that if $x_{,}, y_{,}, z_{,}$ denote the coordinates of the point of contact of the tangent plane to the surface (projective coordinates they might be named, to distinguish them from the tangential coordinates);

* On the Application of a new Analytical Method to the Theory of Curves and Curved Surfaces. London, Parker.

$$\left.\begin{array}{l} n\,x_{,}=L\,\xi-V\,\zeta-W\,\upsilon \\ n\,y_{,}=M\,\upsilon-W\,\xi-U\,\zeta \\ n\,z_{,}=N\,\zeta-U\,\upsilon-V\,\xi \end{array}\right] \quad . \quad . \quad (82).$$

Let $x'\ y'\ z'$ denote the coordinates of the foot of the perpendicular P on the tangent plane; then as $P\cos\lambda = x'$, and $P\,\xi = \cos\lambda$, $x' = P^2\,\xi$; in like manner, $y' = P^2\,\upsilon$, $z' = P^2\,\zeta$: whence

$$\left.\begin{array}{l} n\,(x_{,}-x')=(L-n\,P^2)\,\xi-V\,\zeta-W\,\upsilon \\ n\,(y_{,}-y')=(M-n\,P^2)\,\upsilon-W\,\xi-U\,\zeta \\ n\,(z_{,}-z')=(N-n\,P^2)\,\zeta-U\,\upsilon-V\,\xi \end{array}\right\} \quad . \quad (83).$$

Now writing Δ for the distance measured along the tangent plane between the foot of the perpendicular upon it from the centre, and the point of contact of this tangent plane, $x_{,}-x'$, $y_{,}-y'$, $z_{,}-z'$, are the projections of Δ upon the three coordinate axes. It is also evident that $(x_{,}\,y_{,}\,z_{,})$, $(x'\,y'\,z')$, and $(0, 0, 0,)$ are the projective coordinates of the three angles of the right-angled triangle whose vertex is at the centre and whose base is Δ.

It may easily be shown, and we may therefore assume, that the orthogonal projections of the area of this triangle upon the coordinate planes of $x\,y$, $y\,z$, and $x\,z$, are

$$[y'(x_{,}-x')-x'(y_{,}-y')],\ [z'(y_{,}-y')-y'(z_{,}-z')],\ \text{and}\ [x'(z_{,}-z')-z'(x_{,}-x')],$$

respectively.

If we substitute in these expressions the values of the projective coordinates, which may be deduced from (83), writing Γ for the area of this triangle, and $\Gamma\,l'$, $\Gamma\,m'$, $\Gamma\,n'$, for its projections on the coordinate planes of $y\,z$, $x\,z$, and $x\,y$, (l', m', n' being the direction cosines which a normal to the plane of Γ makes with the axes of x, y, z, respectively) we shall have

$$\left.\begin{array}{l} n\,\Gamma\,l'=P^2[(M-N)\zeta\upsilon-(W\,\zeta-V\,\upsilon)\,\xi-U(\zeta^2-\upsilon^2)] \\ n\,\Gamma\,m'=P^2[(N-L)\xi\zeta-(U\,\xi-W\zeta)\upsilon-V(\xi^2-\zeta^2)] \\ n\,\Gamma\,n'=P^2[(L-M)\xi\upsilon-(V\,\upsilon-U\,\xi)\zeta-W(\upsilon^2-\xi^2)] \end{array}\right\} \quad (84).$$

We shall discover the dynamical illustrations of these expressions further on.

XL. To determine the axes of figure of the ellipsoid.

It is manifest that whenever the distance Δ between the foot of the perpendicular from the centre on the tangent plane, and the point of contact of this tangent plane with the surface, vanishes, that the radius through the point of contact becomes also a perpendicular to the tangent plane, and therefore one of the axes of the surface. When $\Delta=0$, its projections on the coordinate axes vanish, or $x_{,}-x'=0$, $y_{,}-y'=0$, $z_{,}-z'=0$; (83) then becomes, putting n, as we evidently may do, equal to 1,

$$\left.\begin{aligned} 0&=(\text{L}-\text{P}^2)\xi-\text{V}\zeta-\text{W}\upsilon \\ 0&=(\text{M}-\text{P}^2)\upsilon-\text{W}\xi-\text{U}\zeta \\ 0&=(\text{N}-\text{P}^2)\zeta-\text{U}\upsilon-\text{V}\xi \end{aligned}\right\} \quad . \quad . \quad (85).$$

from these equations eliminating the quantities ξ, υ, ζ, we get the following cubic equation in P^2,

$$(\text{L}-\text{P}^2)(\text{M}-\text{P}^2)(\text{N}-\text{P}^2)-\text{U}^2(\text{L}-\text{P}^2)-\text{V}^2(\text{M}-\text{P}^2)-\text{W}^2(\text{N}-\text{P}^2)-2\,\text{UVW}=0. \quad (86).$$

The roots of this equation are the three semiaxes squared of the ellipsoid.

We need not here stop to show that the three roots of this cubic equation are real, as the proposition has already been established in various ways. The following is a group of symmetrical formulæ for determining the position of any one of these axes in space when its magnitude is determined:

Let R^2 be one of the roots of the cubic equation, or the square of one of the semiaxes; let $\text{L}-\text{R}^2=\text{Q}$, $\text{M}-\text{R}^2=\text{Q}'$, $\text{N}-\text{R}^2=\text{Q}''$; also, let λ, μ, ν, be the angles which this axe R makes with the axes of coordinates; then $\cos\lambda=\text{P}\xi$, $\cos\mu=\text{P}\upsilon$, $\cos\nu=\text{P}\zeta$.

Resuming equations (85), and introducing the given value R^2 of P^2,

$$\left.\begin{aligned} \text{Q}\,\xi-\text{V}\,\zeta-\text{W}\upsilon&=0 \\ \text{Q}'\upsilon-\text{W}\xi-\text{U}\,\zeta&=0 \\ \text{Q}''\zeta-\text{U}\upsilon-\text{V}\,\xi&=0 \end{aligned}\right\} \quad . \quad . \quad (87).$$

Combining the first of these equations with the second, and eliminating υ,

$\dfrac{\xi}{\zeta}=\dfrac{\text{V}\text{Q}'+\text{UW}}{\text{QQ}'-\text{W}^2}$; combining the second with the third,

and eliminating υ,

$\frac{\xi}{\zeta}=\frac{Q'Q''-U^2}{VQ'+UW}$; multiplying the two latter,

$$\frac{\xi^2}{\zeta^2}=\frac{\cos^2\lambda}{\cos^2\nu}=\frac{Q'Q''-U^2}{Q'Q-W^2}. \quad \text{Similarly } \frac{\upsilon^2}{\xi^2}=\frac{\cos^2\mu}{\cos^2\lambda}=\frac{Q''Q-V^2}{Q''Q'-U^2}$$

whence adding

$$\cos^2\lambda=\frac{Q''Q'-U^2}{(Q''Q'-U^2)+(QQ''-V^2)+(Q'Q-W^2)} \qquad (88)$$

and similar expressions for $\cos^2\mu$, $\cos^2\nu$.

We may express these formulæ in a more compact notation as follows:

If we take the first derivative of (86), we shall find it to consist of three members. Substituting for P^2 one of its values, R^2 suppose, the resulting expression may be written

$\Upsilon+\Phi+\Omega$, and the last formula becomes

$$\cos^2\lambda=\frac{\Upsilon}{\Upsilon+\Phi+\Omega}; \text{ also } \cos^2\mu=\frac{\Phi}{\Upsilon+\Phi+\Omega}, \cos^2\nu=\frac{\Omega}{\Upsilon+\Phi+\Omega}.$$

XLI. Before we pass from this portion of the subject it may not be out of place to show what the geometrical representations are of the constants in the equation

$$A\xi^2+A'\upsilon^2+A''\zeta^2+2B\upsilon\zeta+2B'\xi\zeta+2B''\zeta\upsilon=1. \qquad (89).$$

Let the tangent plane be parallel to the plane of xy, then $\xi=0$, $\upsilon=0$, and the last equation becomes $A''\zeta^2=1$; but in this case $\frac{1}{\zeta}$ is the distance from the origin to the tangent plane touching the surface parallel to the plane of xy, whence A'' denotes the square of this distance; or A, A' A'' are the squares of the perpendiculars let fall from the centre on the tangent planes parallel to the coordinate planes of yz, xz, and xy respectively.

Let us continue the supposition that the tangent plane remains parallel to the plane of xy, let $x_, y_, z_,$ be the coordinates of the point of contact, and x' y' z' the coordinates of the foot of the perpendicular on the tangent plane; then referring to (83) we shall find, as the tangent plane is parallel to the plane of xy, that the perpendicular on the tangent plane coincides with the

axis of z; whence $x' = 0$, $y = 0$, $\xi = 0$, $\upsilon = 0$ and $A''\zeta = 1$. Making these substitutions in the following equations,

$$\begin{aligned} x_{,} &= A\,\xi + B'\zeta + B''\upsilon, \\ y_{,} &= A'\upsilon + B''\xi + B\zeta, \\ z_{,} &= A''\zeta + B\upsilon + B'\xi, \end{aligned}$$

we get

$$x_{,} = \frac{B'}{\sqrt{A''}},\ y_{,} = \frac{B}{\sqrt{A''}},\ \text{whence as } \sqrt{A''} = z_{,}$$

$B = y_{,}z_{,}$, $B' = z_{,}x_{,}$, $B'' = x_{,}y_{,}$.

The attentive reader will not fail to observe that the constants of the tangential equation of a surface of the second order admit of a much simpler geometrical interpretation than the analogous quantities in the projective equation of the same surface.

XLII. In every revolving body there exists an instantaneous axis of rotation, or a line of particles which remain at rest during an instant. Let P be the position of a point in the revolving body at any given time, P′ the position of the point during the next instant. Let the arc PP′ be ds. At the extremities of this arc ds let normal planes be drawn to the curve. If these planes are parallel, the motion is one of rotation round an axe infinitely distant, or the motion is one of translation. If the planes are not parallel let them meet, the right line in which they intersect is the axis of rotation during the indefinitely small time in which the arc PP′ or ds has been described.

This line, the intersection of the normal planes must pass through the fixed point, if there be one in the body, otherwise there would exist in the body a fixed point and a fixed right line, not passing through the point, which would retain the body in a state of rest, contrary to the supposition.

Again, there cannot be, during the same instant, two or more axes of rotation in the body, for two fixed lines are equivalent to three fixed points, which would retain the body in a state of rest.

The same considerations will show that the instantaneous axis of rotation could not possibly be a curve.

The angular velocity of a body is defined to be the arc of a circle whose radius is 1, described in the element of the time, and whose centre is on the axis of rotation.

XLIII. To determine the equations of the instantaneous axis of rotation.

The fixed point being taken as origin, let $x'y'z'$ be the coordinates of the point P, $(x'+dx')$, $(y'+dy')$, $(z'+dz')$ of the point P′. The equation of the normal plane passing through P is

$$x dx' + y dy' + z dz' = x' dx' + y' dy' + z' dz' = 0,$$

since the plane must pass through the origin; hence as $x' dx' + y' dy' + z' dz' = 0$, the point P must move on the surface of a sphere. The equation of the normal plane passing through P′ is

$$x d^2 x' + y d^2 y' + z d^2 z' = 0$$

The equation of the osculating plane passing through the arc ds being

$$\mathrm{A}(x-x') + \mathrm{B}(y-y') + \mathrm{C}(x-z') = 0,$$

we may determine the constants from the consideration that the osculating plane is perpendicular to each of the normal planes. The osculating plane is therefore perpendicular to the intersection of these planes, that is, to the instantaneous axis of rotation.

Let λ, μ, ν be the angles which this line makes with the axes of coordinates, then $\dfrac{\cos\lambda}{\cos\nu} = \dfrac{\mathrm{A}}{\mathrm{C}}$, $\dfrac{\cos\mu}{\cos\nu} = \dfrac{\mathrm{B}}{\mathrm{C}}$;

and the equations of this right line are,

$$\mathrm{A}z - \mathrm{C}x = 0, \quad \mathrm{B}x - \mathrm{A}y = 0, \quad \mathrm{C}y - \mathrm{B}z = 0. \quad (a).$$

Let ω be the angular velocity round the instantaneous axis of rotation;

then $\omega = \dfrac{ds}{\mathrm{R}}$, R being the radius of curvature.

Make $r = \omega \cos\nu$, and as

$$\cos\nu = \frac{\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}, \quad r = \frac{ds}{\mathrm{R}} \; \frac{\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}.$$

Now R (as is shown in treatises on the geometry of three dimensions*,) is equal to $\frac{ds^3}{\sqrt{A^2 + B^2 + C^2}}$;

whence $r = \frac{C}{ds^2}$; in like manner, let $p = \omega \cos \lambda$, $q = \omega \cos \mu$,

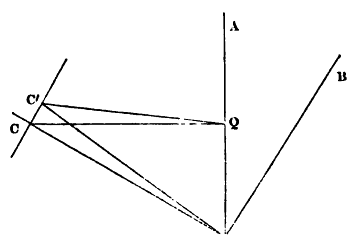
$$\text{then } \quad = \frac{A}{ds^2}, \; q = \frac{B}{ds^2}$$

substituting in (a) those values of A, B, C, we get

$$pz - rx = 0, \; qx - py = 0, \; ry - qz = 0 \quad . \qquad (90)$$

These are the equations of the instantaneous axis of rotation, as we shall show presently from dynamical considerations.

XLIV. The angular velocity round the instantaneous axis being ω, the angle velocity round any other axis which makes the angle θ with the former is $\omega \cos \theta$.



Let O A be the instantaneous axis of rotation, O B an axis which makes the angle θ with the former. Through O let a plane be drawn perpendicular to O B. In this plane assume any point C, with the centre O and radius O G let a sphere be described, and through C let a plane be drawn perpendicular to O A and meeting this line in Q. The point C will move in consequence of this rotation on the circumference of the circle made in the sphere by this plane, and therefore on the surface of the sphere itself. Hence the tangent C C′ is perpendicular as well to the line C O as to C A. Let the angle C Q C′ $= \omega$, the angle C O C′ $= \omega'$, then C C′ $=$ C Q. $\omega =$ O C. ω' and C Q $=$ O C $\cos \theta$, hence

$$\omega' = \omega \cos \theta. \quad . \quad . \quad . \qquad (91).$$

Now as the angular velocities of every other element of the body, round the axes O A, O B, are ω and ω' respectively, during

* *Leroy, Analyse appliquée à la Géométrie des Trois Dimensions*, p. 295.

this instant; it is plain that the angular velocity of every particle of the body round these axes is connected by the relation

$$\omega' = \omega \cos \theta.$$

hence p, q, r, in the last section, are the angular velocities round the axes of x, y, z.

XLV. Let as before $\mathrm{O}x$, $\mathrm{O}y$, $\mathrm{O}z$, be any three rectangular coordinates passing through the fixed point O; X, Y, Z, the velocities of the particle dm of the body resolved along these axes, xyz being the coordinates of the particle dm. These velocities being translated to the origin are there equilibrated by the resistance of the fixed point O; While they generate the moments $(\mathrm{Y}x - \mathrm{X}y)\,d\mathrm{m}$, $(\mathrm{Z}y - \mathrm{Y}z)\,d\mathrm{m}$, $(\mathrm{X}z - \mathrm{Z}x)\,d\mathrm{m}$ in the planes of xy, yz, and zx respectively.

We may conventionally assume that the rotations from x to y, from y to z, and from z to x, shall be taken as positive, and the rotations in any of the opposite directions as negative. Now let ω be the angular velocity round the instantaneous axis of rotation, λ, μ, ν the angles this axis makes with the axes of coordinates, p, q, r the components of the angular velocities along the axes of xyz; so that

$$p = \omega \cos \lambda, \quad q = \omega \cos \mu, \quad r = \omega \cos \nu \quad . \quad . \quad (92).$$

The velocity of the particle dm parallel to the plane of xy is $r\sqrt{x^2+y^2}$ and that resolved along the axes of x and y is $-r\sqrt{x^2+y^2}.\dfrac{y}{\sqrt{x^2+y^2}}$ and $r\sqrt{x^2+y^2}.\dfrac{x}{\sqrt{x^2+y^2}}$; or $-yr$ and xr, in accordance with the conventional agreement as to the signs of rotation in the coordinate planes; whence

The velocities parallel to the axes of x and y, are $-yr$ and xr;
" " " of y and z, are $-zp$ and yp;
" " " of z and x, are $-xq$ and zq;

whence $\mathrm{X} = zq - yr$, $\mathrm{Y} = xr - zp$, $\mathrm{Z} = yp - xq$.

and these velocities translated to the origin generate the moments,

$$\left.\begin{aligned} \mathrm{Y}x - \mathrm{X}y &= (xr - zp)\,x - (zq - yr)\,y, \text{ in the plane of } xy. \\ \mathrm{Z}y - \mathrm{Y}z &= (yp - xq)\,y - (xr - zp)\,z, \text{ in the plane of } yz. \\ \mathrm{X}z - \mathrm{Z}x &= (zq - yr)\,z - (yp - xq)\,x, \text{ in the plane of } xz. \end{aligned}\right\} \quad . \quad (93).$$

We may determine the position of that group of particles, if any, in the body which at the given instant are at rest, by making $X = 0$, $Q = 0$, $Z = 0$. These conditions are satisfied by making $xr - zp = 0$, $zq - yr = 0$, $yp - xq = 0$.

These, it is hardly necessary to observe, are the equations of a right line passing through the origin; equations which we have already found (91) from geometrical considerations.

XLVI. If we extend to the whole mass, the velocities found for the single particle dm in the preceding section, we must integrate the expressions for those velocities. Introducing the notation adopted in (76), we find

$$\left.\begin{aligned} \int (Xz - Zx)\,dm &= Mq - Wp - Ur \\ \int (Yx - Xy)\,dm &= Nr - Uq - Vp \\ \int (Zy - Yz)\,dm &= Lp - Vr - Wq \end{aligned}\right\} \quad . \quad . \quad (94).$$

Now as the impressed couple, or the resultant of all the impressed couples, must, by the principle of D'Alembert, be equivalent to the effective moments, if we make this impressed couple K, and l, m, n, its direction cosines,

$$\left.\begin{aligned} Kl &= Lp - Vr - Wq \\ Km &= Mq - Wp - Ur \\ Kn &= Nr - Uq - Vp \end{aligned}\right\} \quad . \quad . \quad . \quad (95).$$

When the principal axes are the axes of coordinates, $U = 0$, $V = 0$, $W = 0$, and we get the well-known equations,

$$Kl = Lp, \quad Km = Mq, \quad Kn = Nr.$$

Hence the components of the angular velocity round the principal axes are equal to the components of the impressed couple at right angles to these axes, divided by the moments of inertia about them, or

$$p = \frac{Kl}{L}, \; q = \frac{Km}{M}, \; r = \frac{Kn}{N}. \quad . \quad . \quad . \quad (96).$$

XLVII. If we compare together the formulæ given in (82) and (95), we shall make the second members identical by assuming

$$p = f\xi,\ q = f\upsilon,\ r = f\zeta,\ f \text{ being a constant}; \quad . \quad (97).$$

whence $\omega^2 = f^2(\xi^2 + \upsilon^2 + \zeta^2) = \frac{f^2}{\text{P}^2}$, or the angular velocity is inversely proportional to the perpendicular on the tangent plane.

Resuming the equations (82) and (95), introducing also the relations established in (97), we obtain

$\text{K}l = \text{L}p - \text{V}r - \text{W}q = f(\text{L}\xi - \text{V}\zeta - \text{W}\upsilon) = f\text{n}\,x_{,}$, or

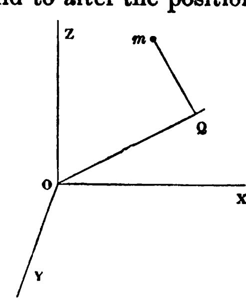
$\text{K}l = f\text{n}x_{,}$, in like manner $\text{K}m = f\text{n}y_{,}$, $\text{K}n = f\text{n}z_{,}$, whence

$$\text{K}^2 = f^2\text{n}^2(x_{,}^2 + y_{,}^2 + z_{,}^2) = f^2\text{n}^2k^2. \quad . \quad . \quad (98).$$

Now $x_{,}\ y_{,}\ z_{,}$ are the coordinates of the point of contact of the tangent plane; whence we infer that k, the radius vector drawn from the centre to the point of contact of the instantaneous plane of rotation, is constant during the motion.

From the relations of (97), it also follows that if through the fixed point we draw any three rectangular axes in the body, the angular velocities round these axes are always inversely proportional to the segments of those axes cut off by the instantaneous plane of rotation; or, in other words, the symbols ξ, υ, ζ, the tangential coordinates of the instantaneous plane of rotation, will denote the components of angular *slowness* round those axes.

XLVIII. Resulting from the rotation of the body, there arises a new class of forces, which in general tend to alter the position of the axes of rotation of the body. They are known as the centrifugal forces. When translated to the origin they generate a couple, whose magnitude and position we are now to determine.



Let OQ be the instantaneous axes of rotation, $\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}$, the cosines of

the angles it makes with the axes. xyz are the coordinates of the particle dm. The centrifugal force which acts on this particle dm, is equal to the square of the velocity divided by the radius; that is, $=\frac{\omega^2\,\overline{Qm}^2}{Qm}=\omega^2.\,Qm$, and this force, as it acts in the direction of Qm, may be resolved into the forces $\omega^2(x-x')$, $\omega^2(y-y')$, $\omega^2(z-z')$, respectively, parallel to the axes of x, y, and z. $x'\,y'\,z'$ are the coordinates of the point Q. Now

$$OQ=Om\cos mOQ=\frac{px+qy+rz}{\omega},$$ we also have

$$x'=OQ\cos QOx=OQ\frac{p}{\omega}=\frac{(px+qy+rz)}{\omega^2}p,$$ or

$\omega^2x'=(px+qy+rz)p$, but $\omega^2x=x(p^2+q^2+r^2)$; whence

$$\left.\begin{aligned}\omega^2(x-x')&=q(qx-py)+r(rx-pz)=X'\\ \omega^2(y-y')&=r(ry-qz)+p(py-qx)=Y'\\ \omega^2(z-z')&=p(pz-rx)+q(qz-ry)=Z'\end{aligned}\right\}\quad(99).$$

From these equations we obtain

$$Y'x-X'y=pq(y^2-x^2)+yx(p^2-q^2)+rz(py-qx);$$

or extending this expression to the whole mass,

$$\int(Y'x-Xy')\,dm=pq\int[(y^2+z^2)-(x^2+z^2)]\,dm+(p^2-q^2)\int yx\,dm+pr\int zy\,dm-qr\int zx\,dm$$

Writing analogous formulæ for the other axes, making Gl', Gm', Gn', equal respectively to $\int dm\,(Z'y-Y'x)$, $\int dm\,(X'z-Z'x)$, $\int dm\,(Y'x-X'y)$, and using the notation established in (76), we get

$$\left.\begin{aligned}G\,l'&=(M-N)qr+(Vq-Wr)p+U(q^2-r^2).\\ Gm'&=(N-L)pr+(Wr-Up)q+V(r^2-p^2).\\ G\,n'&=(L-M)pq+(Up-Vq)r+W(p^2-q^2).\end{aligned}\right\}\ (100).$$

When the principal axes coincide with the axes of coordinates, $U=0$, $V=0$, $W=0$, and the formulæ become

$$Gl'=(M-N)qr,\ Gm'=(N-L)pr,\ Gn'=(L-M)pq.\quad(101).$$

When one of the axes of coordinates, that of z suppose,

coincides with the instantaneous axis of rotation, we have $p=0$, $q=0$, $r=\omega$, and (100) becomes

$$\mathrm{G}l'=-\mathrm{U}\omega^2,\ \mathrm{G}m'=\mathrm{V}\omega^2,\ \mathrm{G}n'=0. \qquad . \quad (102).$$

XLIX. If we multiply the first of (100) by $\frac{p}{\omega}$, the second by $\frac{q}{\omega}$, the third by $\frac{r}{\omega}$, and add the results, the sum will be zero, or

$$\frac{\mathrm{G}}{\omega}\{l'p+m'q+n'r\}=0; \quad . \quad . \quad (103);$$

whence it follows that, *the plane of the centrifugal couple always passes through the instantaneous axis of rotation.*

L. Multiply together line by line the groups in (95) and (100), add the results, the sum will be cypher; or

$$\mathrm{K}\,\mathrm{G}\{ll'+mm'+nn'\}=0 \quad . \quad . \quad . \quad (104)$$

Whence we infer that the *planes of the impressed and centrifugal couples are always at right angles to each other.*

LI. If we compare the formula (84) with (100), we shall find the second members identical, if we assume as, in (97),

$$p=f\xi,\ q=f\upsilon,\ r=f\zeta;\ \text{whence}$$

$$\mathrm{G}=\Gamma\,\mathrm{n}\,\omega^2 \quad . \quad . \quad . \quad (105).$$

We may hence infer that the triangle between the radius vector, the tangent plane, and the perpendicular on it from the centre, coincides in position with the plane of the centrifugal couple. The centrifugal couple is also equal to the *centrifugal triangle* multiplied by the mass and the square of the angular velocity.

The reader will not fail to have observed the ease and simplicity with which the properties of the ellipsoid, treated generally, without reference to the principal axes, by the method of tangential coordinates, may be used to illustrate and establish the corresponding states of a body in motion round a fixed point. The subsequent investigations might in most cases have been

discussed with the same generality and facility; but as the principles of this new analytical geometry, the method of tangential coordinates, is probably but little known, it may be more satisfactory to conduct our investigations on principles universally admitted. To simplify the results, we shall adopt a particular system of coordinates which will render the formulæ much more manageable. If we choose the principal axes as axes of coordinates, $\text{U} = 0$, $\text{V} = 0$, $\text{W} = 0$, and our investigations will therefore be very much simplified.

LII. Let a, b, c, be the three semiaxes of the ellipsoid in the order of magnitude, L, M, N the moments of inertia about the coinciding principal axes of the body. We may assume the squares of the semiaxes of the ellipsoid proportional to the moments of inertia round these axes, so that

$$a^2\text{n} = \text{L}, \quad b^2\text{n} = \text{M}, \quad c^2\text{n} = \text{N}. \quad . \quad . \quad . \qquad (106)$$

n being a constant depending on the mass and constitution of the body.

This ellipsoid we shall call the ellipsoid of moments.

LIII. Introducing these transformations and simplifications, (77), (95), and (100) become

$$\text{H} = \text{n}\{a^2\cos^2\lambda + b^2\cos^2\mu + \cos^2\nu\} \quad . \quad . \quad . \quad . \qquad (107)$$
$$\text{K}l = \text{n}a^2p, \; \text{K}m = \text{n}b^2q, \; \text{K}n = \text{n}c^2r \; . \quad . \quad . \quad . \qquad (108)$$
$$\text{G}l' = \text{n}(b^2-c^2)\,qr, \; \text{G}m' = \text{n}(c^2-a^2)\,pr, \; \text{G}n' = \text{n}(a^2-b^2)\,pq. \quad (109).$$

In formula (107) it is evident that the part within the braces is the expression for the square of a perpendicular from the centre on a tangent plane to the ellipsoid. Let this perpendicular be P, and (107) will become

$$\text{H} = \text{n}\text{P}^2 \quad . \quad . \quad . \quad . \quad . \qquad (110).$$

Square the terms of (108), add them, and multiply by ω^2, we get the result

$$\text{K}\omega^2 = \text{n}^2\;[a^4p^2 + b^4q^2 + c^4r^2]\;(p^2 + q^2 + r^2)$$

$$\text{also as } \omega\cos\lambda = p, \;\; \omega\cos\mu = q, \;\; \omega\cos\nu = r,$$

$$\mathrm{H}^2\omega^4 = \mathrm{n}^2\ [a^2p^2 + b^2q^2 + c^2r^2]^2.$$

Whence we obtain

$$\mathrm{G}^2 = \mathrm{K}^2\omega^2 - \mathrm{H}^2\omega^4 \quad . \quad . \quad . \quad . \quad (111);$$

a formula which gives the value of the centrifugal couple in terms of the impressed couple, the moment of inertia, and the angular velocity round the instantaneous axis of rotation.

LIV. Assume the impressed couple $\mathrm{K} = \mathrm{n}fk$, k being the semidiameter of the ellipsoid perpendicular to the plane of K. The product fk is of course constant; it will be shown presently that f and k are each constant.

As the axes of coordinates are the principal axes,

$$p = \frac{\mathrm{K}l}{\mathrm{L}}, \; q = \frac{\mathrm{K}m}{\mathrm{M}}, \; r = \frac{\mathrm{K}n}{\mathrm{N}}. \text{ See (96).}$$

Let xyz be the coordinates of the vertex of k, then

$$l = \frac{x}{k}, \; m = \frac{y}{k}, \; n = \frac{z}{k}, \mathrm{K} = \mathrm{n}fk, \; \mathrm{L} = \mathrm{n}\,a^2, \; \mathrm{M} = \mathrm{n}b^2, \; \mathrm{N} = \mathrm{n}\,c^2;$$

$$\text{whence } p = \frac{fx}{a^2}, \; q = \frac{fy}{b^2}, \; r = \frac{fz}{c^2} \quad . \quad . \quad . \quad (112.)$$

Squaring those values and adding, $(p^2 + q^2 + r^2) = \omega^2$

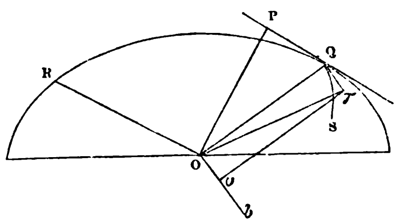
$$= f^2\left\{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right\} = \frac{f^2}{\mathrm{P}^2} \quad . \quad . \quad (113).$$

The cosines of the angles which this perpendicular makes with the axes are, $\frac{\mathrm{P}x}{a^2}\ \frac{\mathrm{P}y}{b^2}\ \frac{\mathrm{P}z}{c^2}$, while the cosines of the angles which the instantaneous axis of rotation makes with the same axes are $\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}$; but $p = \frac{fx}{a^2}$ and $\omega = \frac{f}{\mathrm{P}}$; whence $\frac{p}{\omega} = \frac{\mathrm{P}x}{a^2}$, similarly $\frac{q}{\omega} = \frac{\mathrm{P}y}{b^2}$ $\frac{r}{\omega} = \frac{\mathrm{P}z}{c^2}$;

we may therefore infer that,

The instantaneous axis of rotation coincides with the perpendicular from the centre on the tangent plane drawn through the vertex of k the axe of the impressed couple. The angular velocity round this axe is inversely proportional to this perpendicular.

LV. During the whole period of rotation, the semidiameter k of the ellipsoid, perpendicular to the plane of the impressed couple, is constant.



Through any point Q on the surface of an ellipsoid let a tangent plane be drawn, and through the centre a plane parallel to it. Let a concentric sphere be described through the point Q, intersecting the surface of the ellipsoid in the curve of double curvature Qs. To this curve, let a tangent Qτ be drawn at the point Q, and through this tangent, let a diametral plane be drawn intersecting in the right line Ob, the diametral plane ROb parallel to the tangent plane through Q.

Then QO, Ob are the semiaxes of the plane section of the surface QOb. Let OQ $= k$, O$b = u$. Let fall from O a perpendicular OP on the tangent plane QPτ. This line will also be perpendicular to the parallel diametral plane ObR, and therefore to every line in this plane, and therefore to the line Ob. Now the tangent line Qτ, as it is on the tangent plane to the ellipsoid, and passes through point Q, must be a tangent to the plane section of the ellipsoid passing through it, and as it is besides a tangent to a curve drawn upon the surface of the sphere, it must be at right angles to the radius of the sphere OQ; hence OQτ is a right angle, and therefore OQ must be a semiaxis of the section OQτ; because when a tangent to a conic section is perpendicular to the diameter passing through the point of contact, this diameter must be an axis of the section. Now as the parallel planes QPτ, ORb are cut by the plane QOτ, Ob is parallel to Qτ and consequently at right angles to OQ. Hence OQ, Ob are the semiaxes of the section OQτ.

Since Ob is perpendicular to OP as well as to OQ, it is per-

pendicular to the plane of OPQ, which passes through OP, OQ, that is, to the plane of the centrifugal couple. Whence we are led to infer that the semiaxes k and u of the diametral section of the ellipsoid, whose plane passes through the tangent to the curve of double curvature in which the ellipsoid and sphere intersect, are perpendicular to the axes of the impressed and centrifugal couples K and G respectively.

Assume a point v on the line Ob, so that Ov may be to k, as the centrifugal couple G is to the impressed couple K. The diagonal Oτ of this instantaneous rectangle will represent, as well in magnitude as in direction, the axis of the resultant couple at the end of the first instant. During this instant accordingly, the vertex of the axe of the impressed couple will have travelled on the surface of the ellipsoid, as also on the surface of the concentric sphere whose radius is k. It follows therefore that at the end of the first instant, the vertex of the axe of the resultant couple will be found on the curve of double curvature in which the ellipsoid and sphere intersect. The same proof will hold for the second and for every succeeding instant, whence k always continues invariable. Now the impressed couple K was assumed in (LIV.), equal to nfk, but as n and k are each constant f must likewise be constant.

If to fix our ideas we take the plane of K horizontal, and k therefore vertical, we may infer that the rotatory motion of the body will be such, that its representative ellipsoid will bring all its semidiameters which are equal to k successively into a vertical position, and therefore the surface of the representative ellipsoid will always pass through a point fixed in space.

LVI. It was shown in (113) that the angular velocity ω was equal to $\frac{f}{\text{P}}$; and as f is constant, the angular velocity round the instantaneous axis of rotation varies inversely as P the perpendicular let fall from the centre on the instantaneous plane of rotation.

LVII. *The angular velocity $\varkappa$ round the axis of the impressed couple is constant during the motion.*

Let θ be the angle between k and P. Then $\cos\theta = \frac{\text{P}}{k}$; now $x = \omega\cos\theta$, as shown in (91), and $\omega = \frac{f}{\text{P}}$, whence $x = \frac{f}{\text{P}} \cdot \frac{\text{P}}{k} = \frac{f}{k}$, but f and k are each constant, or $x = \frac{f}{k} = \text{constant}$. (114).

LVIII. *The magnitude of the centrifugal couple* G, *varies as the tangent of the angle between the axis of the impressed moment, and the instantaneous axis of rotation.*

Resume the equation given in (111), $\text{G}^2 = \text{K}^2\omega^2 - \text{H}\omega^4$. Write for K, H and ω their values given in (LIV.), (107), and (113), namely, $\text{K} = \text{n}fk$, $\text{H} = \text{n}\text{P}^2$, and $\omega = \frac{f}{\text{P}}$. We have also,

$\tan\theta = \frac{\sqrt{k^2 - \text{P}^2}}{\text{P}}$, $x = \frac{f}{k}$, whence

$$\text{G} = \text{K}\, x \tan\theta \quad . \quad . \quad . \quad . \quad (115).$$

LIX. It will be evident on inspection, that the indefinitely small portion Ov (see fig., page 64.) of the line Ob parallel to the tangent drawn at Q, to the section of the ellipsoid whose semi-axes are k and u, and which is equal to Qτ, may be taken as the element of the arc of the spherical curve traced out by the vertex of k during the element of the time dt. Writing $\frac{ds}{dt}$ for this element Ov, and referring to (LIV.) we have the ratio Ov : k :: G : K,

$$\text{or } \text{O}v = \frac{ds}{dt} = \frac{\text{G}k}{\text{K}}, \text{ but } \text{G} = \text{K}x\tan\theta, \text{ and } f = xk.$$

$$\text{Whence } \frac{ds}{dt} = f\tan\theta \quad . \quad . \quad . \quad (116).$$

Now $\frac{ds}{dt}$ is the velocity with which the curve of double curvature passes through Q, the fixed point in space. We thence deduce, that the velocity with which the pole of the impressed couple passes along this curve, or the velocity with which the

curve passes through the fixed pole, varies as the tangent of the angle between the axis of the impressed couple and the instantaneous axis of rotation.

LX. To find the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, or of the velocities of the pole of the impressed couple in the direction of the principal axes of the body.

We have $\frac{dz}{ds} = \frac{\frac{dz}{dt}}{\frac{ds}{dt}}$, and $\frac{ds}{dt} = f \tan\theta$, whence $\frac{dz}{dt} = \frac{dz}{ds} f \tan\theta$,

and $\frac{ds^2}{dz^2} = 1 + \frac{dx^2}{dz^2} + \frac{dy^2}{dz^2}$. Now (xyz) is a point on the surface of the ellipsoid of moments, as also on that of a concentric sphere whose radius is k. The equations of these surfaces are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ and } x^2 + y^2 + z^2 = k^2 \quad . \quad . \quad (117).$$

Eliminating y and x successively, and then differentiating, we find

$$\frac{dx}{dz} = \frac{z}{x}\frac{a^2}{c^2}\left(\frac{b^2-c^2}{a^2-b^2}\right), \quad \frac{dy}{dz} = \frac{z}{y}\frac{b^2}{c^2}\left(\frac{a^2-c^2}{a^2-b^2}\right) \quad . \quad (118).$$

Whence $\frac{ds^2}{dz^2} = \frac{a^4(b^2-c^2)^2y^2z^2 + b^4(c^2-a^2)^2x^2z^2 + c^4(a^2-b^2)^2x^2y^2}{c^4(a^2-b^2)^2x^2y^2}$

$$\text{Now } \tan^2\theta = \frac{k^2 - \mathrm{P}^2}{\mathrm{P}^2} = k^2\left\{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right\} - \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right\}^2$$

and $k^2 = x^2 + y^2 + z^2$; hence

$$\tan^2\theta = \frac{a^4(b^2-c^2)^2y^2z^2 + b^4(c^2-a^2)^2z^2x^2 + c^4(a^2-b^2)^2x^2y^2}{a^4b^4c^4} \quad (119);$$

or eliminating x and y by (117),

$$\tan^2\theta = \frac{(a^2-c^2)(b^2-c^2)k^2z^2 - c^4(a^2-k^2)(b^2-k^2)}{a^2b^2c^4} \quad . \quad (120).$$

Making the substitutions suggested by these equations, we obtain

$$\frac{dz}{dt}=\frac{f(a^2-b^2)xy}{a^2b^2},\ \frac{dx}{dt}=\frac{f(b^2-c^2)yz}{\quad},\ \frac{dy}{dt}=\frac{f(c^2-a^2)xz}{a^2c^2}.\ (121).*$$

LXI. *The axis of rotation due to the centrifugal forces lies in the plane of the impressed couple.*

Let ω' be the angular velocity round the axis of rotation due to the centrifugal couple, p', q', r' its components round the principal axes. Then as the angular velocity round any principal axe is equal to the couple which produces the motion resolved at right angles to this axe, and divided by the corresponding moment of inertia,

$$p'=\frac{\text{G}\frac{dx}{ds}}{\text{L}},\ \text{now } \text{G}=\text{K}\,\varkappa\tan\theta,\ \text{K}=\text{n}fk,\ \text{L}=\text{n}a^2,$$

$$\text{and } \frac{dx}{ds}=\frac{\frac{dx}{dt}}{\frac{ds}{dt}}=\frac{\frac{dx}{dt}}{f\tan\theta},\ \text{whence } p'=f\frac{\frac{dx}{dt}}{a^2}$$

making corresponding substitutions for q' and r', we have

$$p'=f\frac{\frac{dx}{dt}}{a^2},\ q'=f\frac{\frac{dy}{dt}}{b^2},\ r'=f\frac{\frac{dz}{dt}}{c^2}\quad .\quad .\ (122).$$

Now the cosines of the angles which this axis of rotation

---

* When the axis of the impressed moment very nearly coincides with one of the principal axes, that of c suppose, the differential equations of motion may easily be deduced.

In this case as x and y are each very small, their product xy may be neglected;

now $p=\frac{fx}{a^2}$, $q=\frac{fy}{b^2}$, $r=\frac{fz}{c^2}$, and $\frac{dr}{dt}=\frac{f}{c^2}\frac{dz}{dt}=\frac{f^2(a^2-b^2)}{a^2b^2c^2}xy=0$. hence r is constant, equal to n suppose. We also have

$$\frac{dp}{dt}=\frac{f}{a^2}\frac{dx}{dt}=\frac{f^2(b^2-c^2)yz}{a^2b^2c^2},\ \text{but } y=\frac{b^2q}{f},\ z=\frac{c^2n}{f}$$

whence $\frac{dp}{dt}=\frac{f^2}{a^2}\frac{(b^2-c^2)}{b^2c^2}\frac{b^2c^2}{f^2}nq=\frac{b^2-c^2}{a^2}nq$, or writing $\text{A}=\text{n}a^2$, $\text{B}=\text{n}b^2$, $\text{C}=\text{n}c^2$;

$$\text{A}\frac{dp}{dt}+(\text{C}-\text{B})\,nq=0.\quad \text{Similarly}$$

$$\text{B}\frac{dq}{dt}+(\text{C}-\text{A})\,np=0.$$

These are the equations deduced by Poisson for this particular case. (*Traité de Mécanique*, tom. ii. p. 159.)

makes with the axes of coordinates are $\frac{p'}{\omega'}$, $\frac{q'}{\omega'}$, $\frac{r'}{\omega'}$, and the cosines of the angles which the axis k makes with the same axes are $\frac{x}{k}$, $\frac{y}{k}$, $\frac{z}{k}$. If we denote the angle between the axis k of the impressed couple, and P′ the instantaneous axis of rotation due to the centrifugal couple by $k\hat{O}P'$,

$$\cos k\hat{O}P' = \frac{1}{k\omega'}(p'x + q'y + r'z) = \frac{f}{k\omega'}\left(\frac{x\frac{dx}{dt}}{a^2} + \frac{y\frac{dy}{dt}}{b^2} + \frac{z\frac{dz}{dt}}{c^2}\right) = 0 \qquad (123).$$

Since the part within the brackets is the differential of the equation of the ellipsoid.

We infer, therefore, that not only is the axis of the centrifugal couple contained in the plane of the impressed couple, but that the axis round which the centrifugal couple would give the body a tendency to revolve, lies in the same plane also.*

LXII. Through the vertex of u the axis of the centrifugal couple, let a tangent plane to the ellipsoid be drawn. The perpendicular from the centre on this tangent plane, is the instantaneous axis of rotation due to the centrifugal couple.

Let $x'\,y'\,z'$ be the coordinates of the vertex of u; l', m', n', the cosines of the angles it makes with the axes. λ', μ', ν', the angles which the instantaneous axis of rotation due to the centrifugal couple makes with the same axes. Then as u is perpendicular as well to P as to K,

$$\frac{l'x}{k} + \frac{m'y}{k} + \frac{n'z}{k} = 0. \quad \mathrm{P}\left\{\frac{l'x}{a^2} + \frac{m'y}{b^2} + \frac{n'z}{c^2}\right\} = 0. \qquad (a).$$

---

* To determine the angular velocity when L = M, or, using Poisson's notation, when A = B.

As $r = \frac{fz}{c^2}$, $\frac{dr}{dt} = \frac{f}{c^2}\,\frac{dz}{dt} = \frac{f^2(a^2 - b^2)}{a^2 b^2 c^2}\,xy = 0$, since $a^2 = b^2$. Hence r is constant $= n$.

Now $\omega^2 = p^2 + q^2 + r^2 = n^2 + \frac{f^2}{a^4}(x^2 + y^2)$. Let $\frac{x^2 + y^2}{k^2} = \sin^2 \epsilon$;

then $\omega^2 = n^2 + \frac{f^2 k^2}{a^4}\sin^2 \epsilon$. We have $\mathrm{K} = nfk$, $\mathrm{A} = na^2$; whence $\omega^2 = n^2 + \frac{\mathrm{K}^2}{\mathrm{A}^2}\sin^2 \epsilon$.

The expression given by Poisson, *Traité de Mécanique*, p. 159.

Eliminating from those equations m' and l' successively,

$$\frac{l'}{n'}=\frac{a^2}{c^2}\frac{z}{x}\left(\frac{b^2-c^2}{a^2-b^2}\right) \text{ (b).} \quad \frac{m'}{n'}=\frac{b^2 z}{c^2 y}\left(\frac{a^2-c^2}{a^2-b^2}\right) \text{ (c).}$$

$$\text{Now } \frac{\cos\lambda'}{\cos\nu}=\frac{\frac{p'x'}{a^2}}{\frac{p'z'}{c^2}}=\frac{c^2x'}{a^2z'}, \quad \frac{\cos\mu'}{\cos\nu'}=\frac{c^2y'}{b^2z'}$$

and $\dfrac{l'}{n'}=\dfrac{\frac{x'}{u}}{\frac{z'}{u}}=\dfrac{x'}{z'},\ \dfrac{m'}{n'}=\dfrac{y'}{z'}$; whence

$\dfrac{\cos\lambda'}{\cos\nu'}=\dfrac{c^2}{a^2}\dfrac{l'}{n'},\ \dfrac{\cos\mu'}{\cos\nu'}=\dfrac{c^2}{b^2}\dfrac{m'}{n'}$; substituting for $\dfrac{l'}{n'}, \dfrac{m'}{n'}$, their values given in the preceding equations, and reducing, we find

$$\cos^2\nu'=\frac{(a^2-b^2)^2x^2y^2}{(a^2-b^2)^2x^2y^2+(b^2-c^2)^2y^2z^2+(c^2-a^2)^2z^2x^2}. \quad (124).$$

We may find analogous expressions for $\cos\lambda'$ and $\cos\mu'$.

Introducing the terms $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}$, by the help of (121),

$$\cos^2\nu'=\frac{a^4b^4\left(\frac{dz}{dt}\right)^2}{a^4b^4\left(\frac{dz}{dt}\right)^2+b^4c^4\left(\frac{dx}{dt}\right)^2+c^4a^4\left(\frac{dy}{dt}\right)^2}. \quad (125).$$

Now the cosine of the angle which the axis due to G makes with the axis of z is $\dfrac{r'}{\omega'}$; writing for r' and ω' their values given in (122),

$$\frac{r'}{\omega'}=\frac{a^4b^4\left(\frac{dz}{dt}\right)^2}{a^4b^4\left(\frac{dz}{dt}\right)^2+b^4c^4\left(\frac{dx}{dt}\right)^2+a^4c^4\left(\frac{dy}{dt}\right)^2}. \quad (126).$$

Whence, comparing (125) with (126),

$$\frac{r'}{\omega'}=\cos\nu'; \text{ in like manner } \frac{p'}{\omega'}=\cos\lambda',\ \frac{q'}{\omega'}=\cos\mu';$$

or, *The perpendicular let fall from the centre on the tangent plane, drawn through the vertex of the axis of the centrifugal couple, coincides with the instantaneous axis of rotation due to this couple.*

The perpendicular P' is therefore in the plane of the impressed couple.

LXIII. To find the component of the angular velocity ω due to the centrifugal couple resolved along the instantaneous axis of rotation.

Let δ be the angle between the axes of the rotations due to the impressed and centrifugal couples. Then

$$\cos\delta = \frac{pp' + qq' + rr'}{\omega\,\omega'},$$ or substituting the values

of $\omega, p, q, r, \omega', p', q', r'$, as given in (112) and (122), we shall have

$$\omega' \cos\delta = Pf\left\{\frac{x}{a^4}\,\frac{dx}{dt} + \frac{y}{b^4}\,\frac{dy}{dt} + \frac{z}{c^4}\,\frac{dz}{dt}\right\}.$$

Now the part within the brackets is the differential of $\dfrac{1}{2\,P^2}$,

whence $\omega' \cos\delta = \dfrac{-f}{P^2}\,\dfrac{dP}{dt} = f\dfrac{d}{dt}\left(\dfrac{1}{P}\right)$, but as $\omega = \dfrac{f}{P}$,

$$\frac{d\omega}{dt} = f\frac{d}{dt}\left(\frac{1}{P}\right), \text{ whence } \frac{d\omega}{dt} = \omega' \cos\delta. \qquad . \quad (127).$$

Or, *The increment of the angular velocity round the instantaneous axis of rotation, is due to the component of the angular velocity arising from the centrifugal couple, and resolved along the axis.*

LXIV. To investigate expressions for the lengths of u and P'.

As u makes angles with the coordinate axes whose cosines are $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}$, and is besides a semidiameter of the surface,

$$\frac{1}{u^2}=\frac{\left(\frac{dx}{ds}\right)^2}{a^2}+\frac{\left(\frac{dy}{ds}\right)^2}{b^2}+\frac{\left(\frac{dz}{ds}\right)^2}{c^2}.$$

Now $\frac{dx}{ds}=\frac{dx}{dt}\,\frac{dt}{ds}$, $\frac{dy}{ds}=\frac{dy}{dt}\,\frac{dt}{ds}$, $\frac{dz}{ds}=\frac{dz}{dt}\,\frac{dt}{ds}$, and $\frac{ds}{dt}=f\tan\theta$, as in (116).

Whence $$\frac{1}{u^2}=\frac{b^2c^2\left(\frac{dx}{dt}\right)^2+a^2c^2\left(\frac{dy}{dt}\right)^2+a^2b^2\left(\frac{dz}{dt}\right)^2}{a^2b^2c^2f^2\tan^2\theta} \qquad (128).$$

Again, as $P'^2=a^2\cos^2\lambda'+b^2\cos^2\mu'+c^2\cos^2\nu'$, we shall have, putting for $\cos\lambda'$, $\cos\mu'$, $\cos\nu'$ their values as given in (124),

$$P'^2=\frac{a^2b^2c^2\left[a^2b^2\left(\frac{dz}{dt}\right)^2+a^2c^2\left(\frac{dy}{dt}\right)^2+b^2c^2\left(\frac{dx^2}{dt}\right)\right]}{a^4b^4\left(\frac{dz}{dt}\right)^2+b^4c^4\left(\frac{dx}{dt}\right)^2+a^4c^4\left(\frac{dy}{dt}\right)^2} \qquad (129).$$

LXV. If we combine (116), (128), and (129), we shall find

$$\frac{\left(\frac{ds}{dt}\right)^2}{P'^2u^2}=\frac{\left(\frac{dx}{dt}\right)^2}{a^4}+\frac{\left(\frac{dy}{dt}\right)^2}{b^4}+\frac{\left(\frac{dz}{dt}\right)^2}{c^4};$$

but $p'=f\frac{\left(\frac{dx}{dt}\right)}{a^2}$, $q'=f\frac{\left(\frac{dy}{dt}\right)^2}{b^2}$, $r'=f\frac{\left(\frac{dz}{dt}\right)}{c^2}$ as shown in (122).

Whence $$\frac{\left(\frac{ds}{dt}\right)^2}{P'^2u^2}=\frac{\omega'^2}{f^2} \quad . \quad . \quad . \quad . \qquad (130).$$

And as $\frac{ds}{dt}=\frac{Gk}{K}$ see (LIX.), and $\omega=\frac{f}{P}$, we shall have

$$\frac{\omega'}{\omega}=\frac{GPk}{KP'u} \quad . \quad . \quad . \quad . \quad . \qquad (131).$$

LXVI. To investigate an expression for the angle ρ, between the axes of rotation due to the impressed and centrifugal couples.

The cosines of the angles which the axes of rotation make with the axes of coordinates are

$$\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}, \frac{p'}{\omega'}, \frac{q'}{\omega'}, \frac{r'}{\omega'}, \text{ whence } \cos\rho = \frac{pp'+qq'+rr'}{\omega\,\omega'};$$

$$\text{now } p = \frac{fx}{a^2}, \text{ and } p' = \frac{f\left(\frac{dx}{dt}\right)}{a^2} = \frac{f^2\,(b^2-c^2)\,yz}{a^2\,b^2\,c^2};$$

whence $pp' = \frac{f^3xyz}{a^2\,b^2\,c^2}\left(\frac{b^2-c^2}{a^2}\right)$. Finding similar expressions for

$$qq' \text{ and } rr', \quad \omega\,\omega'\cos\rho = \frac{f^3xyz}{a^2\,b^2\,c^2}\left\{\frac{a^2-b^2}{c^2} + \frac{b^2-c^2}{a^2} + \frac{c^2-a^2}{b^2}\right\}$$

$$\text{but } \left[\frac{a^2-b^2}{c^2} + \frac{b^2-c^2}{a^2} + \frac{c^2-a^2}{b^2}\right] = \frac{(a^2-b^2)\,(b^2-c^2)\,(a^2-c^2)}{a^2\,b^2\,c^2}$$

$$\text{whence } \omega\,\omega'\cos\rho = \frac{f^3xyz\,(a^2-b^2)\,(b^2-c^2)\,(a^2-c^2)}{a^4\,b^4\,c^4} \qquad (132).$$

The values of ω and ω' are given in (113) and (130).

This formula shows that *whenever any two of the axes of the ellipsoid of moments are equal, or whenever the axis of the impressed couple happens to lie in one of the principal planes of the ellipsoid, the angle between the axes of rotation due to the impressed and centrifugal couples is a right angle.*

SECTION III.

To determine the cones described by the axes of the impressed and centrifugal couples, as also by the axes of rotation due to those couples; in other words, to investigate the loci of k, P, u, and P′ referred to the principal axes of the body during the motion, is the object of the present section.

LXVII. To find the locus of k.

The equation of the cone whose vertex is at the centre, and

which passes through the curve in which the ellipsoid of moments, and the invariable sphere whose radius is k, intersect, may easily be shown to be,

$$\left(\frac{1}{k^2}-\frac{1}{a^2}\right)x^2+\left(\frac{1}{k^2}-\frac{1}{b^2}\right)y^2+\left(\frac{1}{k^2}-\frac{1}{c^2}\right)z^2=0. \quad (133).$$

The equation of a cone of the second degree, whose axes coincide with those of the ellipsoid.

This cone and the spherical conic section which constitutes its base will repeatedly present themselves in the course of the following pages; it may therefore be proper to denote them by some appropriate name.

As the side of this cone is constant, being the axis of the impressed couple, it may with propriety be named the *invariable cone*, and the spherical conic may be termed the *invariable spherical ellipse*.

LXVIII. To investigate the nature of the surface described by P the instantaneous axis of rotation.

λ, μ, ν, being the angles which P makes with the axes,

$$\frac{\cos\lambda}{\cos\nu}=\frac{c^2x}{a^2z} \quad \frac{\cos\mu}{\cos\nu}=\frac{c^2y}{b^2z}.$$

We have also the equations of the ellipsoid and sphere,

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $x^2+y^2+z^2=k^2$; eliminating x, y, z, we get

$$a^4\left(\frac{1}{a^2}-\frac{1}{k^2}\right)\cos^2\lambda+b^4\left(\frac{1}{b^2}-\frac{1}{k^2}\right)\cos^2\mu+c^4\left(\frac{1}{c^2}-\frac{1}{k^2}\right)\cos^2\nu=0.$$

Let xyz be the coordinates of any point on the surface of the cone at the distance R from the origin, then $\cos\lambda=\frac{x}{R}$, $\cos\mu=\frac{y}{R}$, $\cos\nu=\frac{z}{R}$, and the equation of the cone becomes,

$$a^4\left(\frac{1}{a^2}-\frac{1}{k^2}\right)x^2+b^4\left(\frac{1}{b^2}-\frac{1}{k^2}\right)y^2+c^4\left(\frac{1}{c^2}-\frac{1}{k^2}\right)z^2=0 \quad (134).$$

The equation of a cone which is also of the second degree.

As this cone too will frequently recur, we may name it the *cone of rotation.*

LXIX. To determine the equation of the cone described by the axis u of the centrifugal couple.

Let $x'y'z'$ be the coordinates of a point on the axis u of the centrifugal couple; then

$$\frac{x'}{z'}=\frac{\frac{dx}{ds}}{\frac{dz}{ds}}=\frac{a^2}{c^2}\left(\frac{b^2-c^2}{a^2-b^2}\right)\frac{z}{x},\quad \frac{y'}{z'}=\frac{\frac{dy}{ds}}{\frac{dx}{ds}}=\frac{b^2}{c^2}\left(\frac{a^2-c^2}{a^2-b^2}\right)\frac{z}{y}.\quad \text{See (118).}$$

From these equations and the equations of the ellipsoid and sphere, eliminating x, y, z, we find

$$a^2(a^2-k^2)(b^2-c^2)^2y^2z^2+b^2(b^2-k^2)(a^2-c^2)^2x^2z^2+c^2(c^2-k^2)(a^2-b^2)^2x^2y^2=0 \quad (135),$$

an equation of the fourth degree.*

LXX. To determine the equation of the cone described in the body by P′ the axis of rotation due to the centrifugal couple.

---

* It may not be out of place to show that the equations of the invariable cone, and of the cone of rotation given in (LXVII.) and (LXVIII.) are equivalent to the equations of the same cones given by Poisson in his *Traité de Mécanique* (tom. ii. pp. 151, 152.). To show this, assume the equation of the vis viva given at page 140 of the same volume, $h=\text{A}p^2+\text{B}q^2+\text{C}r^2$. Now $\text{A}=\text{n}a^2$, $p=\frac{fx}{a^2}$, whence $\text{A}p^2=\text{n}f^2\frac{x^2}{a^2}$; finding similar values for $\text{B}q^2$ and $\text{C}r^2$, we obtain

$$h=\text{A}p^2+\text{B}q^2+\text{C}r=\text{n}f^2\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right\}=\text{n}f^2;\ \text{we also have}$$

$\text{A}=\text{n}a^2$, $\text{B}=\text{n}b^2$, $\text{C}=\text{n}c^2$;

$$k'=\text{K}=\text{n}fk,\ \text{hence}\ k'^2=\text{n}.\ \text{n}f^2k^2=\frac{\text{A}}{a^2}.\ h.\ k^2,\ \text{or}\ \frac{a^2}{k^2}=\frac{\text{A}h}{k'^2}$$

but the coefficient $\left(\frac{1}{a^2}-\frac{1}{k^2}\right)$ may be written $\frac{1}{a^2}\left(1-\frac{a^2}{k^2}\right)=\frac{\text{n}}{\text{A}}\left(1-\frac{\text{A}h}{k'^2}\right)=$ $\frac{\text{n}}{k'^2}\left(\frac{k'^2-\text{A}h}{\text{A}}\right)$; making similar substitutions for the other coefficients and dividing by $\frac{\text{n}}{k'^2}$, we get

$$\left(\frac{k'^2-\text{A}h}{\text{A}}\right)x^2+\left(\frac{k'^2-\text{B}h}{\text{B}}\right)y^2+\left(\frac{k'^2-\text{C}h}{\text{C}}\right)z^2=0.$$

In the same way, (134) may be transformed into,

$$(k'^2-\text{A}h)\,x^2+(k'^2-\text{B}h)\,y^2+(k'^2-\text{C}h)\,z^2=0.$$

These are the equations given by Poisson.

The axis P′ makes with the axes of coordinates the angles λ, μ', ν'. Let $x'\ y'\ z'$ be the coordinates of a point on the surface of this cone; then

$$\frac{x'}{z'}=\frac{\cos\lambda'}{\cos\nu'}=\frac{z}{x}\left(\frac{b^2-c^2}{a^2-b^2}\right),\quad \frac{y'}{z'}=\frac{z}{y}\left(\frac{a^2-c^2}{a^2-b^2}\right).\quad \text{See (124).}$$

Eliminating $x\ y\ z$ from these equations, as also from those of the ellipsoid and sphere,

$$\frac{(a^2-k^2)}{a^2}(b^2-c^2)^2 y^2 z^2+\frac{(b^2-k^2)}{b^2}(a^2-c^2)^2 x^2 z^2+\frac{(c^2-k^2)}{c^2}(a^2-b^2)^2 x^2 y^2=0 \quad (136),$$

which is also an equation of the fourth degree.

LXXI. The circular sections of the invariable cone coincide in position with the circular sections of the ellipsoid.

It is a property of surfaces of the second order*, that if in

---

* Let $Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+Cx+C'y+C''z=1$, be the equation of a surface of the second degree, referred to rectangular axes. Let the surface now be referred to a new system of rectangular coordinates, such that the plane of $x'y'$ shall be parallel to one of the umbilical tangent planes, or to one system of circular sections of the surface. If in this transformed equation we make $z'=0$, we shall obtain the equation of a circle referred to rectangular axes, if the roots are real. The equation being that of a circle, we thence derive two conditions, the equality of the coefficients of the squares of the variables, and the evanescence of the coefficient of the rectangle $x'y'$. Let θ be the angle between the axes of z and z'. If we take the intersection of the plane of xy with the plane of one of the circular sections as the axis of x', ψ being the angle between the axes of x and x', we shall have, by the known transformations of coordinates, and putting $z'=0$,

$$x=\cos\psi x'+\cos\theta\sin\psi y',\quad y=-\sin\psi x'+\cos\theta\cos\psi y',\quad z=-\sin\theta y'.$$

Substituting these values of $x\,y\,z$ in the given equation, the resulting equation in x' and y' is that of the conic section in which the plane of $x'y'$ intersects the given surface. As this section must be a circle, we get the two conditions

$$[(A-A'')\cos^2\psi+(A'-A'')\sin^2\psi-2B''\sin\psi\cos\psi]\tan^2\theta+2[B\cos\psi+B'\sin\psi]\tan\theta$$
$$=4B''\sin\psi\cos\psi-(A-A')(\cos^2\psi-\sin^2\psi),\text{ and}$$

$$\tan\theta=\frac{B''(\cos^2\psi-\sin^2\psi)+(A-A')\sin\psi\cos\psi}{B'\cos\psi-B\sin\psi}.$$

From those equations eliminating $\tan\theta$, we shall obtain a resulting equation of condition in ψ, whose coefficients will be functions of $(A-A')$, $(A-A'')$, $(A'-A'')$, B, B', B''.

As the coefficients of the squares of the variables do not enter the coefficients of the resulting equation, but the differences of those coefficients only, it follows that two surfaces of the second order whose equations are of the form,

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy\ \&c.=1.$$

$$(A+h)x^2+(A'+h)y^2+(A''+h)z^2+2Byz+2B'xz+2B''xy\ \&c.=1,$$

will have the planes of their circular sections parallel.

two such surfaces referred to the same or parallel axes, the coefficients of the squares of the corresponding variables differ all by the same quantity, the circular sections of any two such surfaces are parallel.

Now the coefficients of the squares of the variables in the equation of the ellipsoid are $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, and the coefficients of the equation of the cone are $\frac{1}{a^2}-\frac{1}{k^2}$, $\frac{1}{b^2}-\frac{1}{k^2}$, $\frac{1}{c^2}-\frac{1}{k^2}$, of which the constant difference is $\frac{1}{k^2}$.

LXXII. There are some general properties of rotatory motion, such as the principles of the conservation of areas, the conservation of living forces, &c., which may with much simplicity be established.

Resuming the equation (74) and multiplying by ω^2 we get

$\mathrm{H}\omega^2 = \int d\mathrm{m}\,[\mathrm{R}\sin\theta.\,\omega]^2$. The integral being extended to the whole mass of the body. Now $\mathrm{R}\sin\theta\,\omega$ is the velocity of the particle dm. The above integral, therefore denotes the sum of all the elementary particles of the body multiplied each into the square of its velocity. This is termed the vis viva of the body.

In (110) it was shown, that $\mathrm{H}=\mathrm{n}\mathrm{P}^2$, and $\omega=\frac{f}{\mathrm{P}}$; whence $\mathrm{H}\omega^2=\mathrm{n}f^2$, or the vis viva of the body is constant, since n and f are constant.

Let the vis viva of the body be denoted by F, we shall have

$$\mathrm{F}=\text{constant} \quad . \quad . \quad . \quad (137).$$

Multiply the tangential equation of the ellipsoid of moments given in (81) by f^2, then

$$\mathrm{n}f^2=\mathrm{L}f^2\xi^2+\mathrm{M}f^2\upsilon^2+\mathrm{N}f^2\zeta^2-2\,\mathrm{U}f^2\upsilon\zeta-2\,\mathrm{V}f^2\xi\zeta-2\,\mathrm{W}f^2\xi\upsilon.$$

In (97), it was shown that $p=f\xi$, $q=f\upsilon$, $r=f\zeta$, whence

$$\mathrm{F}=\mathrm{L}p^2+\mathrm{M}q^2+\mathrm{N}r^2-2\,\mathrm{U}qr-2\,\mathrm{V}pr-2pq \quad . \quad (138),$$

which is the equation of the vis viva in its most general form. When we take the principal axes as axes of coordinates,

$$\text{U}=0,\ \text{V}=0,\ \text{W}=0, \text{ or } \text{F}=\text{L}p^2+\text{M}q^2+\text{N}r^2 \quad . \qquad (139);$$

the form in which the equation of the vis viva is usually exhibited.

LXXIII. If we square the equations given in (95), and add the results

$$\begin{aligned}\text{K}^2 = (\text{L}^2+\text{V}^2+\text{W}^2)p^2 + (\text{M}^2+\text{W}^2+\text{U}^2)q^2 + (\text{N}^2+\text{U}^2+\text{V}^2)r^2 \qquad (140).\\ -2\,[\text{U}\,(\text{M}+\text{N})-\text{V}\text{W}]\,qr - 2\,[\text{V}\,(\text{N}+\text{L})-\text{W}\text{U}]\,pr - 2\,[\text{W}\,(\text{M}+\text{L})-\text{U}\text{V}]\,pq.\end{aligned}$$

In this equation is contained the principle of the conservation of areas; for $\text{K}\,l$ or its equal $(\text{L}p-\text{V}r-\text{W}q)$ is the sum of the areas described on the plane of yz, multiplied into the particles which describe those areas. Now those areas are projected on the plane of the impressed couple, by multiplying this expression by the cosine of the angle between the planes, that

is, by l or its equal $\dfrac{\text{L}p-\text{V}r-\text{W}q}{\text{K}}$, and therefore $\dfrac{(\text{L}p-\text{V}r-\text{W}q)^2}{\text{K}}$

denotes the sum of the particles of the body multiplied into the areas described by those particles on the plane of yz, and then projected on the plane of the impressed couple. Finding analogous expressions for the two other co-ordinate planes, we get for the sum of all the particles of the body multiplied into the areas which they describe on the plane of the impressed couple,

$$\frac{(\text{L}p-\text{V}r-\text{W}q)^2}{\text{K}}+\frac{(\text{M}q-\text{W}p-\text{U}r)^2}{\text{K}}+\frac{(\text{N}r-\text{U}q-\text{V}p)^2}{\text{K}};$$

but the sum of these expressions must, we know, be equal to K, whence we obtain the formula given above.

When the axes of co-ordinates are the principal axes, $\text{V}=0$, $\text{U}=0$, $\text{W}=0$, and we get the well-known equation,

$$\text{K}^2=\text{L}^2p^2+\text{M}^2q^2+\text{N}^2r^2 \quad . \qquad (141).$$

We may, in a very simple manner, establish the equations

which embody the principles of the vis viva, and the conservation of areas, without using the method of tangential co-ordinates, when we restrict our choice of co-ordinates to the principal axes of the body; for

$$\mathrm{L}=\mathrm{n}\,a^2,\ p=\frac{fx}{a^2}, \text{ as shown in (106) and (112).}$$

Finding similar values for the other analogous quantities and adding

$$\mathrm{L}p^2+\mathrm{M}q^2+\mathrm{N}r^2=\mathrm{n}f^2\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right\}=\mathrm{n}f^2=\mathrm{F},\quad (142).$$

Again, $\mathrm{L}^2p^2+\mathrm{M}^2q^2+\mathrm{N}r^2=\mathrm{n}^2f^2(x^2+y^2+z^2)=\mathrm{n}^2f^2k^2=\mathrm{K}^2$ (143).

LXXIV. Let p', q', r', denote the angular velocities round the principal axes, the components of the angular velocities due to the centrifugal couple; then

$$\mathrm{L}^2p'^2+\mathrm{M}^2q'^2+\mathrm{N}^2r'^2=\mathrm{K}^2\varkappa^2\tan^2\theta\quad .\quad .\quad (144).$$

We have $\mathrm{L}=\mathrm{n}\,a^2,\ p'=f\dfrac{\left(\frac{dx}{dt}\right)}{a^2}$, writing similar expressions for the other analogous quantities,

$$\mathrm{L}^2p'^2+\mathrm{M}^2q'^2+\mathrm{N}^2r'^2=\mathrm{n}^2f^2\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right\}=\mathrm{n}^2f^2\,\frac{ds^2}{dt^2}.$$

Now $\dfrac{ds}{dt}=f\tan\theta$, see (116), and $\dfrac{f}{k}=\varkappa$, as in (114);

whence $\mathrm{L}^2\,p'^2+\mathrm{M}^2\,q^2+\mathrm{N}^2\,r^2=\mathrm{n}^2f^2f^2\tan^2\theta$

$$=\mathrm{n}^2f^2\,k^2.\frac{f^2}{k^2}\tan^2\theta=\mathrm{K}^2\ \varkappa^2\ \tan^2\theta.$$

We may also show that,

$$\mathrm{L}p'^2+\mathrm{M}q'^2+\mathrm{N}r'^2=\mathrm{F}\,\frac{k^2}{u^2}.\ \varkappa^2\tan^2\theta\quad .\quad .\quad (145).$$

LXXV. Using the principles established in the foregoing pages, the reader will find little difficulty in verifying the following theorems:

$$p'l + q'm + r'n = 0 \quad . \quad . \quad . \quad . \quad (146).$$

$$\mathrm{L}p'l + \mathrm{M}q'm + \mathrm{N}r'n = 0 \quad . \quad . \quad . \quad (147).$$

$$\frac{p'l}{\mathrm{L}} + \frac{q'm}{\mathrm{M}} + \frac{r'n}{\mathrm{N}} = \frac{1}{2k}\,\frac{d}{dt}.\,\omega^2 \quad . \quad . \quad . \quad (148).$$

LXXVI. The sum of the squares of the distances of the vertices of the three semiaxes of the ellipsoid of moments from the plan of the impressed couple, divided by the corresponding moments of inertia, is constant during the motion.

Let $x_{\prime}$ be the distance of the vertex of a from the plane of the impressed couple. Then $x_{\prime} = al$, and $l = \frac{x}{k}$, hence $x_{\prime} = \frac{ax}{k}$ and $\mathrm{L} = \mathrm{n}a^2$, or $\frac{x_{\prime}^2}{\mathrm{L}} = \frac{x^2}{\mathrm{n}k^2}$. Whence

$$\frac{x_{\prime}^2}{\mathrm{L}} + \frac{y_{\prime}^2}{\mathrm{M}} + \frac{z_{\prime}^2}{\mathrm{N}} = \frac{1}{\mathrm{n}k^2}(x^2 + y^2 + z^2) = \frac{1}{\mathrm{n}} \quad . \quad (149).$$

LXXVII. The sum of the squares of the distances of the vertices of the three semiaxes of the ellipsoid from the plane of the impressed couple, divided by the squares of the corresponding moments of inertia, is constant during the motion.

As before $x_{\prime}^2 = \frac{a^2x^2}{k^2}$, $\mathrm{L}^2 = \mathrm{n}^2a^4 \therefore \frac{x_{\prime}^2}{\mathrm{L}^2} = \frac{1}{\mathrm{n}^2k^2}\left(\frac{x^2}{a^2}\right)$. Whence

$$\left(\frac{x_{\prime}}{\mathrm{L}}\right)^2 + \left(\frac{y_{\prime}}{\mathrm{M}}\right)^2 + \left(\frac{z_{\prime}}{\mathrm{N}}\right)^2 = \frac{1}{\mathrm{n}^2k^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = \frac{1}{\mathrm{n}^2k^2} \quad (150).$$

LXXVIII. Let tangent planes be drawn to the vertices of a, b, c, the three semiaxes of the ellipsoid, cutting off from the axis of the plane of the impressed couple three segments. The sum of the squares of the reciprocals of those segments will be constant during the motion. Denoting those reciprocals by ξ, υ, ζ, we shall have $\xi^2 + \upsilon^2 + \zeta^2 = \frac{1}{k^2}$, during the motion;

for $a\xi = l = \frac{x}{k}$, or $k\xi = \frac{x}{a}$, hence

$$\xi^2 + \upsilon^2 + \zeta^2 = \frac{1}{k^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = \frac{1}{k^2} \quad . \quad . \quad (151).$$

Again, as $a^2\xi^2 + b^2\upsilon^2 + c^2\zeta^2 = (l^2 + m^2 + n^2) = 1$.

ξ, υ, ζ the reciprocals of the segments cut off from the axis of the plane of the impressed couple by three tangent planes drawn through the vertices of the axes of the surface, may be the segments of the axes of co-ordinates cut off by *any* tangent plane to the ellipsoid.

LXXIX. If through the vertex of k, which is a point fixed in space, a plane be drawn parallel to the plane of the impressed couple, this fixed plane will cut off segments from the axes of the ellipsoid during the motion, the sum of the squares of the reciprocals of which is constant.

Writing ξ, υ, ζ for those reciprocals, we have

$$k\xi = l,\ k\upsilon = m,\ k\zeta = n, \text{ hence } \xi^2 + \upsilon^2 + \zeta^2 = \frac{1}{k^2} \quad (152).$$

SECTION IV.

LXXX. We must now proceed to the investigation of formulæ, by whose aid we may be enabled to determine the position of the body at the end of a given epoch. For this purpose we shall obtain two distinct classes of formulæ, to determine not only relatively to certain fixed lines within the body—the principal axes suppose—the position of certain other lines, but also absolutely the position of those lines themselves in space. This double investigation is necessary, because the locus of a point will vary accordingly as we choose the axes of coordinates fixed in space, or varying in position according to some given law. For example, the instantaneous axis of rota-

tion describes on a sphere concentric with the body, and moving along with it a spherical conic, while it describes on a concentric sphere fixed in space, a spiral which undulates continually between two small parallel circles of the sphere.

Again, under certain conditions the same right line may describe in the body a plane, or on the moving sphere a great circle, while it describes in absolute space a sort of spiral cone, or on the surface of the fixed sphere a spiral, approaching very nearly to the loxodromic or rumb line.

We have hitherto assumed k as lying between the mean and least semiaxes of the ellipsoid, or $a^2 > b^2 > k^2 > c^2$. Should we require to consider the case when k lies between the greatest and mean semiaxes of the ellipsoid, the formulæ will be most easily modified so as to embrace this hypothesis also, by taking in that case c as the greatest semiaxe, and b the mean semiaxe as before, or $a^2 < b^2 < k^2 < c^2$. While on the former supposition the binomials $a^2 - b^2$, $a^2 - c^2$, $b^2 - c^2$, $a^2 - k^2$, $b^2 - k^2$, $k^2 - c^2$, are all positive, on the latter they will all be negative. Now in the formulæ which we shall have to deal with in the remaining portion of this treatise, these binomials occur generally in pairs, connected either by multiplication or division. It will result, therefore, that no effective change of sign will generally take place, whether we suppose k to lie between the greatest and mean semiaxes, or between the mean and the least. The case where k is equal to the mean axe will require a separate investigation. When the body is a solid of revolution we cannot take N equal to L or M, or c equal to a or b, because we suppose c to be the greatest or the least of the three semiaxes. The only hypothesis, not inconsistent with previous assumptions, is L = M, or $a = b$; and this is the assumption generally made, when the case of a solid of revolution is considered.

Resuming the equation (121),

$$\frac{dt}{dz} = \frac{a^2 b^2}{f(a^2 - b^2)\,xy},$$

If we agree to take $\dfrac{dt}{dz}$ with the positive sign when $a > b$, we must attach the negative sign whence $a < b$.

To integrate this equation, we must express x and y in terms z. This we can easily do, by eliminating x and y alternately from the equations of the ellipsoid of moments and the concentric sphere. We hence find

$$x=\frac{a\sqrt{(b^2-c^2)\,z^2-c^2\,(b^2-k^2)}}{c\sqrt{a^2-b^2}},\quad y=\frac{b\sqrt{c^2\,(a^2-k^2)-(a^2-c^2)\,z^2}}{c\sqrt{a^2-b^2}},$$

Making these substitutions, the last equation becomes

$$\frac{dt}{dz}=\frac{abc^2}{f\sqrt{[(b^2-c^2)z^2-c^2(b^2-k^2)]\,[c^2(a^2-k^2)-(a^2-c^2)z^2]}}\text{*}\quad(153).$$

To facilitate the integration of this equation, assume

$$z^2=\frac{c^2\,(a^2-k^2)\,(b^2-k^2)}{(a^2-k^2)\,(b^2-c^2)\cos^2\phi+(b^2-k^2)\,(a^2-c^2)\sin^2\phi}\quad.\quad(154).$$

Substituting the value of z derived from this equation in (153), and integrating,

$$t=\frac{\pm\,abc}{f\sqrt{(a^2-k^2)\,(b^2-c^2)}}\int\frac{d\phi}{\sqrt{1-\left\{\frac{(a^2-b^2)\,(k^2-c^2)}{(a^2-k^2)\,(b^2-c^2)}\right\}\sin^2\phi}}\,(155).$$

An elliptic integral of the first order.

LXXXI. The modulus of this function is the sine of the semifocal angle of the invariable cone.

---

* If we assume the relations established in the note at page (75),

$A=\mathrm{n}a^2$, $B=\mathrm{n}b^2$, $C=\mathrm{n}c^2$, $h=\mathrm{n}f^2$, $k'=\mathrm{n}fk$, $r=\frac{fz}{c^2}$, $\frac{dz}{dr}=\frac{c^2}{f}$, and by the help of these relations eliminate from (153) the quantities a, b, c, f, z, k, we shall obtain the resulting equation

$$dt=\frac{\pm\sqrt{AB}.\;C\,dr}{[k'^2-Bh+(B-C)\,C\,r^2]^{\frac{1}{2}}[Ah-k'^2+(C-A)\,C\,r^2]^{\frac{1}{2}}};$$

the expression which Poisson arrives at, *Traité de Mécanique*, tom. ii. p. 140.

Resuming the equation of this cone given in (133), and writing α and β for its principal angles,

$$\tan^2\alpha=\frac{b^2(k^2-c^2)}{c^2(b^2-k^2)},\ \tan^2\beta=\frac{a^2(k^2-c^2)}{c^2(a^2-k^2)}\quad . \qquad (156).$$

Now ϵ being the semifocal angle of the cone, $\cos\epsilon=\dfrac{\cos\alpha}{\cos\beta}$

see (I), or $\sin^2\epsilon=\dfrac{\cos^2\beta-\cos^2\alpha}{\cos^2\beta}=\dfrac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}$, and

$$\cos^2\epsilon=\frac{(a^2-c^2)(b^2-k^2)}{(a^2-k^2)(b^2-c^2)},\ \text{or}\ \sec\alpha\cos\epsilon=\frac{k\sqrt{(a^2-c^2)(b^2-c^2)}}{c\sqrt{(a^2-k^2)(b^2-c^2)}}$$

and the coefficient of the elliptic integral in (155)

$$\frac{abc}{f\sqrt{(a^2-k^2)(b^2-c^2)}},\ \text{may now be written}\ \frac{abc^2\sec\alpha\cos\epsilon}{fk\sqrt{(a^2-c^2)(b^2-c^2)}}.$$

In (114) it was shown that $f=k\varkappa$; introducing this relation into the preceding coefficient, and making

$$i=\varkappa\cos\alpha\,\frac{k^2}{c^2}\sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2b^2}},\quad .\quad .\ (157)$$

(155) may now be written

$$it=\cos\epsilon\int\frac{d\phi}{\sqrt{1-\sin^2\epsilon\sin^2\phi}}\quad .\quad .\ (157^*).$$

In (55) it was shown that the arc σ of a spherical parabola whose principal arcs $\bar{\alpha}$ and $\bar{\beta}$ are given by the equations

$\tan^2\bar{\alpha}=\dfrac{1+\sin\gamma}{1-\sin\gamma},\ \tan^2\bar{\beta}=\dfrac{2\sin\gamma}{1-\sin\gamma}$, may be represented by an elliptic integral of the first order, or

$$\sigma=\sin\gamma\int\frac{d\phi}{\sqrt{1-\cos^2\gamma\sin^2\phi}}+\tan^{-1}\left\{\frac{\sin\gamma\tan\phi}{\sqrt{1-\cos^2\gamma\sin^2\phi}}\right\}$$

writing ς for the circular arc, we get the simple formula

$$it=\sigma-\varsigma\quad .\quad .\quad .\quad .\ (158).$$

In this case, $\tan^2\bar{\alpha}=\frac{1+\cos\varepsilon}{1-\cos\varepsilon}=\cot^2\frac{1}{2}\varepsilon$, or $2\bar{\alpha}+\varepsilon=\pi$.

$2\bar{\alpha}$ and ε are therefore supplemental.

When ε vanishes, $\bar{\alpha}=\frac{\pi}{2}$, $\bar{\beta}=\frac{\pi}{2}$, or the spherical parabola becomes a great circle of the sphere.

When the moment N of the body is very nearly equal to L or M, c^2 must very nearly be equal to a^2 or b^2, and the coefficient i becomes indefinitely small.

LXXXII. It may easily be shown that the amplitude φ assumed in (154) is the eccentric anomaly of the vertex of k, the axis of the impressed couple. Let a and b be the semiaxes of the plane ellipse, the intersection of the invariable cone with a plane which touches the sphere, whose radius is k. This plane is at right angles to the axis c of the ellipsoid, the internal axe of this cone.

Let the plane which passes through the axis c and k cut the plane of the ellipse in the semidiameter R, making the angle ψ with the axis a of the ellipse. Then as $\mathrm{a}=k\tan\alpha$, $\mathrm{b}=k\tan\beta$, and ρ being the angle which k makes with the axis of z, $\mathrm{R}=k\tan\rho$, we shall have

$$\cos^2\rho=\frac{\dfrac{1}{\tan^2\alpha}+\dfrac{\tan^2\psi}{\tan^2\beta}}{\dfrac{1}{\sin^2\alpha}+\dfrac{\tan^2\psi}{\sin^2\beta}} \qquad \text{as shown in (8).}$$

Let φ' be the eccentric anomaly, then $\tan\varphi'=\frac{\mathrm{a}}{\mathrm{b}}\tan\psi$,

or $\tan\varphi'=\dfrac{\tan\alpha}{\tan\beta}\tan\psi$, and $\cos^2\rho=\dfrac{\cos^2\alpha}{1-\sin^2\varepsilon\sin^2\varphi'}$ (j).

In (154) we assumed

$$\frac{z^2}{k^2}=\frac{\dfrac{c^2}{k^2}\left(\dfrac{b^2-k^2}{b^2-c^2}\right)}{1-\left\{\dfrac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}\right\}\sin^2\phi}, \text{ but } \frac{z}{k}=\cos\rho,\ \frac{c^2(b^2-k^2)}{k^2(b^2-c^2)}=\cos^2\alpha$$

and $\dfrac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}=\sin^2\varepsilon$; comparing this expression with (j) we find $\varphi=\varphi'$,

Or φ is the eccentric anomaly of the vertex of k.

LXXXIII. Resuming the equation established in (157), we may invert the formula, $it = \cos \varepsilon \int \frac{d\phi}{\sqrt{1 - \sin^2 \varepsilon \sin^2 \phi}}$, and express the amplitude in terms of the function. Accordingly let ϕ be a function of it, or $\phi = (it)$,* the parenthesis denoting a function of it. Substituting this value in the value assumed for z in (154), we find the following values of x, y, z.

$$\left.\begin{aligned}
x^2 &= \frac{a^2(b^2-k^2)(k^2-c^2)\sin^2(it)}{(a^2-k^2)(b^2-c^2)\cos^2(it)+(b^2-k^2)(a^2-c^2)\sin^2(it)} \\
y^2 &= \frac{b^2(a^2-k^2)(k^2-c^2)\cos^2(it)}{(a^2-k^2)(b^2-c^2)\cos^2(it)+(b^2-k^2)(a^2-c^2)\sin^2(it)} \\
z^2 &= \frac{c^2(a^2-k^2)(b^2-k^2)}{(a^2-k^2)(b^2-c^2)\cos^2(it)+(b^2-k^2)(a^2-c^2)\sin^2(it)}
\end{aligned}\right\} (159).$$

LXXXIV. We may also express x, y, z in terms of the time and of the constants of the invariable cone. Transforming the expressions given in the formulæ (159), we find

---

* That the assumption here made is allowable, may be shown as follows.

Let $(1 - c^2 \sin^2 \phi)^{-\frac{1}{2}}$ be developed in a series of cosines of multiple arcs, for the successive integral powers of $\sin^2\phi$ may be so developed. Accordingly let

$$\frac{i}{\sqrt{1 - c^2 \sin^2\phi}} = \mathrm{A} + 2\,\mathrm{B} \cos 2\phi + 4\,\mathrm{C} \cos 4\phi + 6\,\mathrm{D} \cos 6\phi \text{ \&c.}$$

Integrating these equivalent expressions, and putting t for $\int \frac{d\phi}{\sqrt{1 - c^2 \sin^2\phi}}$, we get

$it = \mathrm{A}\phi + \mathrm{B} \sin 2\phi + \mathrm{C} \sin 4\phi + \mathrm{D} \sin 6\phi$ . . . &c. now

$$\sin 2\phi = 2\phi - \frac{(2\phi)^3}{1\,2\,3} + \frac{(2\phi)^5}{1\,2\,3\,4\,5} \text{ \&c.}$$

$$\sin 4\phi = 4\phi - \frac{(4\phi)^3}{1\,2\,3} + \frac{(4\phi)^5}{1\,2\,3\,4\,5} \text{ \&c.}$$

$$\sin 6\phi = 6\phi - \frac{(6\phi)^3}{1\,2\,3} \text{ \&c.}$$

Substituting these values of the sines of the multiple arcs of ϕ in the preceding equation,

$it = \alpha\phi + \beta\phi^3 + \gamma\phi^5 +$ &c.;

or, by the inverse method of series,

$\phi = \alpha_{,}[it] + \beta_{,}[it]^3 + \gamma_{,}[it]^5$ &c.; or ϕ may be taken as a function of it, or we may put $\phi = (it)$, as in the text.

$$\left.\begin{array}{l}\dfrac{x^2}{k^2}=\dfrac{\tan^2\beta\sin^2(it)}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)},\ \dfrac{y^2}{k^2}=\dfrac{\tan^2\alpha\cos^2(it)}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)} \\ \dfrac{z^2}{k^2}=\dfrac{1}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)}.\end{array}\right\} \quad (160).$$

From either of those groups of equations we may find the coordinates xyz of the vertex of k the axis of the impressed couple in terms of the time. We can thus determine the particular diameter of the ellipsoid which happens to coincide with the axis of the impressed couple at the end of the time t. And if we suppose the ellipsoid brought into this position, we shall have the inclination of the equator of the body to the plane of the impressed couple. This however is not sufficient to determine completely the position of the body. The body might take any position round this line as an axis, xyz remaining unchanged. We must therefore determine the position of some other fixed line or plane in the body. One of the most obvious is the intersection of the plane of the equator of the body or of the plane of xy with the plane of the impressed couple. The position of this line being ascertained at any epoch, the position of the body will be completely determined.

LXXXV. To determine the value of ω the angular velocity at the end of any given time.

Since $\omega^2=\dfrac{f^2}{P^2}=f^2\left\{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4},\right\}$ substituting for xyz their values given in terms of the time in (160), we find

$$\omega^2=k^2f^2\frac{\left\{\frac{1}{c^4}+\frac{\tan^2\alpha}{b^4}\cos^2(it)+\frac{\tan^2\beta}{a^4}\sin^2(it)\right\}}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)} \quad . \quad . \quad (161).$$

This formula may be simplified as follows.

It was shown in (LXVIII.) that the instantaneous axis of rotation describes a cone of the second degree, whose equation is

$$\left(1-\frac{a^2}{k^2}\right)a^2x^2+\left(1-\frac{b^2}{k^2}\right)b^2y^2+\left(1-\frac{c^2}{k^2}\right)c^2z^2=0.$$

Let α' and β' be the principal angles of this cone. It may easily be shown that

$$\tan^2\alpha' = \frac{c^2(k^2-c^2)}{b^2(b^2-k^2)}, \quad \tan^2\beta' = \frac{c^2(k^2-c^2)}{a^2(a^2-k^2)} \quad . \qquad (162).$$

$$\text{Whence } \tan\alpha' = \frac{c^2}{b^2}\tan\alpha, \ \tan\beta' = \frac{c^2}{a^2}\tan\beta. \quad . \qquad (163).$$

Introducing into the value of ω these functions, we get

$$\omega^2 = \frac{f^2 k^2}{c^4}\left\{\frac{\sec^2\alpha'\cos^2(it) + \sec^2\beta'\sin^2(it)}{\sec^2\alpha\cos^2(it) + \sec^2\beta\sin^2(it)}\right\}^* \quad . \qquad (164).$$

---

* Let the axis of the impressed couple very nearly coincide with one of the principal axes,—that of c suppose,—then k is very nearly equal to c, or to z, and the angular velocity round the axis of z, being given by the equation $r = \frac{fz}{c^2}$, as in (112), $r = \frac{f}{c}$, a constant quantity, which may be put equal to n, or $\kappa = n$.

In this case the invariable cone becoming indefinitely attenuated, $\sec\alpha = 1$, $\sin\epsilon = 0$, and $k = c$ nearly, so that the formula given in LXXXI.

$$t = \frac{c^2}{kf}\,\frac{\sec\alpha}{\sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2 b^2}}} \times \cos\epsilon \int \frac{d\phi}{\sqrt{1-\sin^2\epsilon\sin^2\phi}},$$

may now be written, $nt = \frac{\phi}{\sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2 b^2}}}$. To use the notation adopted by Poisson in the *Traité de Mécanique*, let A, B, C, denote the moments of inertia round the principal axes, then $A = \text{n}a^2$, $B = \text{n}b^2$, $C = \text{n}c^2$,

$$\text{whence } \sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2 b^2}} = \sqrt{\frac{(A-C)(B-C)}{AB}} = \delta,$$

or $n\delta t = \phi$, whence $i = n\delta$.

In (159) we found $x^2 = \frac{a^2(b^2-k^2)(k^2-c^2)\sin^2 it}{(a^2-k^2)(b^2-c^2)\cos^2 it + (b^2-k^2)(a^2-c^2)\sin^2 it}$

Since k^2 is equal to c^2 nearly, let $k^2 = c^2 + \nu^2$, in which ν is a quantity indefinitely small; the above formula may now be written

$$x^2 = \frac{\nu^2 a^2[b^2-c^2-\nu^2]\sin^2 n\delta t}{(a^2-c^2)(b^2-c^2) - \nu^2[(b^2-c^2)\cos^2 n\delta t + (a^2-c^2)\sin^2 n\delta t]}$$

We may also express the components p, q, r of the angular velocity in terms of the time

$$\left.\begin{array}{c} p^2=\dfrac{f^2k^2}{a^4}\left\{\dfrac{\tan^2\beta\sin^2(it)}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)}\right\},\ q^2=\dfrac{f^2k^2}{b^4}\left\{\dfrac{\tan^2\alpha\cos^2(it)}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)}\right\} \\ r^2=\dfrac{f^2k^2}{c^4}\left\{\dfrac{1}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)}\right\} \end{array}\right\}(165).$$

LXXXVI. The angles, which the instantaneous axis of rotation makes with the principal axes, are given by the equations

$$\frac{\cos\lambda}{\cos\nu}=\frac{c^2}{a^2}\frac{x}{z}=\frac{c^2}{a^2}\tan\beta\sin(it),\ \frac{\cos\mu}{\cos\nu}=\frac{c^2y}{b^2z}=\frac{c^2}{b^2}\tan\alpha\cos(it)$$

or as $\tan\alpha'=\dfrac{c^2}{b^2}\tan\alpha$, $\tan\beta'=\dfrac{c^2}{a^2}\tan\beta$, as in (163).

$$\frac{\cos\lambda}{\cos\nu}=\tan\beta'\sin(it),\ \frac{\cos\mu}{\cos\nu}=\tan\alpha'\cos(it)\quad .\quad (166).$$

$$\left.\begin{array}{c} \cos^2\lambda=\dfrac{\tan^2\beta'\sin^2(it)}{\sec^2\alpha'\cos^2(it)+\sec^2\beta'\sin^2(it)};\cos^2\mu=\dfrac{\tan^2\alpha'\cos^2(it)}{\sec^2\alpha'\cos^2(it)+\sec^2\beta'\sin^2(it)}, \\ \cos^2\nu=\dfrac{1}{\sec^2\alpha\cos^2(it)+\sec^2\beta\sin^2(it)} \end{array}\right\}(167).$$

---

or neglecting ν^2 when added to finite quantities,

$x^2=\dfrac{\nu^2a^2(b^2-c^2)\sin^2 n\delta t}{(a^2-c^2)(b^2-c^2)}$. Taking the square root and reducing,

$\dfrac{fx}{a^2}=\dfrac{\nu f\text{N}\sqrt{b^2(b^2-c^2)}.\ \sin n\delta t.}{\sqrt{\text{N}^2a^2b^2(a^2-c^2)(b^2-c^2)}}$. Now assume $\dfrac{\nu f}{\sqrt{\text{N}^2a^2b^2(a^2-c^2)(b^2-c^2)}}=\alpha$,

whence $\dfrac{fx}{a^2}=\alpha\sqrt{\text{B}(\text{B}-\text{C})}\sin(n\delta t+\gamma)$. γ is added, since x and t may be supposed not to vanish together. In like manner, $\dfrac{fy}{b^2}=\alpha\sqrt{\text{A}(\text{A}-\text{C})}\cos(n\delta t+\gamma)$.

In (112) it was shown that $p=\dfrac{fx}{a^2}$, $q=\dfrac{fy}{b^2}$, whence

$$p=\alpha\sqrt{\text{B}(\text{B}-\text{C})}\sin(n\delta t+\gamma),\quad q=\alpha\sqrt{\text{A}(\text{A}-\text{C})}\cos(n\delta t+\gamma).$$

These are the formulæ established by Poisson, on this particular hypothesis, by methods wholly dissimilar. (*Traité de Mécanique*, tom. ii. p. 154.)

When k is absolutely equal to c, $\nu=0$, and therefore $\alpha=0$, or $p=0$, $q=0$, whatever be the value of t. Since $\text{K}=fk\text{N}$, $\text{F}=f^2\text{N}$, we get

$\alpha^2=\dfrac{\text{K}^2-\text{FN}}{\text{LM}(\text{L}-\text{N})(\text{M}-\text{N})}$; or using Poisson's notation, $\alpha^2=\dfrac{k^2-h\text{C}}{\text{AB}(\text{A}-\text{C})(\text{B}-\text{C})}$.

These equations give us the position of the instantaneous axis of rotation with reference to the principal axes, in terms of the time.

LXXXVII. We must now, in order completely to determine the position of the body at the end of the time t, investigate a formula which will enable us to ascertain the position of some other line in the body at the end of the given epoch. We may take the right line in which the equator of the body—the plane of xy suppose—and the plane of the impressed couple intersect.

The angular velocity of the body round the axis k being uniform and equal to $\varkappa$, the angle described on the plane of the impressed moment in the element of the time dt will be $\varkappa\, dt$, or the angle $\varkappa t$ in the time t, measured from a given line in this plane, its intersection with the plane of the equator of the body, or the plane of the axes a, b. But this line will itself have an angular motion on the plane of the impressed moment during the time; this angle may be denoted by ψ. Whence the whole elementary angle will be

$$\frac{d\psi}{dt}+\varkappa. \quad \text{Let this angle be } \frac{d\vartheta}{dt}, \text{ then } \frac{d\psi}{dt}+\varkappa=\frac{d\vartheta}{dt}.$$

Now this elementary angle is the projection on the plane of the impressed moment of the angle on the plane of a, b, over which the projection of the axis k on the plane of a, b, passes in the time dt. Let ρ be the angle between those planes, or the angle between the axes k and z. Then $\cos\rho=\frac{z}{k}$, and the angle of which $\frac{d\vartheta}{dt}$ is the projection is $\frac{k}{z}\,\frac{d\vartheta}{dt}$. Hence the area described on the plane of a, b, by the projection of k upon it, is $\frac{1}{2}(x^2+y^2)\,\frac{k}{z}\,\frac{d\vartheta}{dt}$. This area may also be represented by the expression $\frac{1}{2}\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)$. Equating those expressions for the same elementary area, $(x^2+y^2)\frac{k}{z}\,\frac{d\vartheta}{dt}=\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)$ . . (a).

Now $\frac{dx}{dt}=\frac{f(b^2-c^2)yz}{b^2c^2}$, $\frac{dy}{dt}=\frac{f(c^2-a^2)xz}{a^2c^2}$, as in (121).

Whence

$$y\frac{dx}{dt}-x\frac{dy}{dt}=\frac{fz}{a^2b^2c^2}\{a^2b^2y^2+a^2b^2x^2-a^2c^2y^2-b^2c^2x^2\} \qquad \text{(b).}$$

The equations of the ellipsoid and sphere give

$$a^2b^2y^2+a^2b^2x^2=a^2b^2k^2-a^2b^2z^2;\ b^2c^2x^2+a^2c^2y^2=a^2b^2c^2-a^2b^2z^2.$$

$$\text{Whence } y\frac{dx}{dt}-x\frac{dy}{dt}=fz\left(\frac{k^2-c^2}{z^2}\right) \qquad \text{(c)}$$

And as $x^2+y^2=k^2-z^2$, $\frac{f}{k}=\varkappa$, we at length obtain

$$\frac{d\vartheta}{dt}=\varkappa\left(\frac{k^2-c^2}{c^2}\right)\left(\frac{z^2}{k^2-z^2}\right) \qquad \text{(d)}$$

$$\text{Whence } \frac{d\psi}{dt}+\varkappa=\varkappa\left(\frac{k^2-c^2}{c^2}\right)\left(\frac{z^2}{k^2-z^2}\right) \quad . \quad . \quad (168).$$

To integrate this equation, assume as in (155)

$$\frac{z^2}{c^2}=\frac{(a^2-k^2)(b^2-k^2)}{(a^2-k^2)(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi}.$$

$$\text{Whence } \frac{k^2-c^2}{c^2}\cdot\frac{z^2}{k^2-z^2}=\frac{(a^2-k^2)(b^2-k^2)}{b^2(a^2-k^2)-k^2(a^2-b^2)\sin^2\phi} \qquad \text{(e)}$$

And writing for dt its value given in (154) we obtain by integration

$$\psi=-\varkappa t+\frac{(b^2-k^2)ac}{bk\sqrt{(a^2-k^2)(b^2-c^2)}}\int\frac{d\phi}{\left[1-\frac{k^2}{b^2}\frac{(a^2-b^2)}{(a^2-k^2)}\sin^2\phi\right]\sqrt{1-\sin^2\epsilon\sin^2\phi}}$$

Now e being the eccentricity of the plane base of the cone, the locus of the axis of the impressed couple, (156) gives

$$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{k^2(a^2-b^2)}{b^2(a^2-k^2)} \quad . \quad . \quad (169).$$

$$\text{We find also } \frac{(b^2-k^2)ac}{bk\sqrt{(a^2-k^2)(b^2-c^2)}}=\pm\frac{\tan\beta}{\tan\alpha}\cos\alpha \quad (170).$$

Taking the negative sign when $b > a$.

Introducing those transformations, the last equation becomes

$$\psi = -\varkappa t \pm \frac{\tan\beta}{\tan\alpha} \cos\alpha \int \frac{d\varphi}{[1 - e^2 \sin^2\varphi] \sqrt{1 - \sin^2\varepsilon \sin^2\varphi}} \quad (171).$$

If we now turn to the formula given in (24), we shall find that this elliptic integral is the analytical expression for an arc of the spherical ellipse, supplemental to the one whose principal arcs are α and β; supplemental in this case, therefore, to the invariable conic. Writing σ for this arc, we get the simple relation

$$\psi = -\varkappa t \pm \sigma \quad . \quad . \quad (172).$$

We may hence infer that the line of the nodes describes an angle which is made up of two parts: one of those parts is a circular arc increasing uniformly with the time; the other σ is an arc of the spherical ellipse, which is the base of the cone supplemental to the invariable cone. Now, as the axis of the impressed couple is always a side of the invariable cone, the plane of the impressed couple will always be a tangent plane to the supplemental cone; and it may easily be shown that the line of contact of the plane of the impressed couple with this cone is always at right angles to the line of the nodes.

It follows, therefore, that this line in the time t will describe the angle $\varkappa t \mp \sigma$.

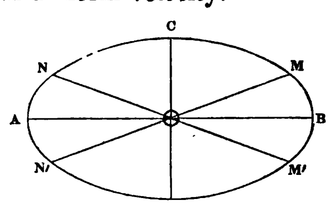
The angle $\psi = \varkappa t \mp \sigma$, we may conceive to be thus described. Let the supplemental cone be conceived to roll on the plane of the impressed couple with such a velocity that the axis of the conjugate tangent plane may describe the invariable cone with the velocity given in (116). Let, moreover, the invariable plane be conceived to revolve uniformly round its axis. We shall then have a perfect idea of the rotatory motion of a body revolving round a fixed point, free from the action of accelerating forces. In this manner it is shown that the most general motion of a body round a fixed point may be reduced to that of a cone which rolls without sliding with a certain variable velocity on a plane whose axis is fixed, while this plane rotates round its axis with a certain uniform velocity.

This cone is always given, and may be determined as follows:

The circular sections of the invariable cone coincide with the

circular sections of the ellipsoid of moments, see (LXXI.). Whence the cyclic axes of the ellipsoid, or the diameters perpendicular to the planes of those sections, will be the focal lines of the supplemental cone. As the invariable plane is always a tangent plane to this cone, we have elements sufficient given to determine it. For when the two focals of a cone and a tangent plane to it are given, we may determine it, just as we may a conic section, when its foci and a tangent to it are given.

LXXXVIII. From these considerations it follows that we may altogether dispense with the ellipsoid of moments, and say, that if two right lines are drawn through the fixed point of the body, in the plane of the greatest and least moments of inertia, making angles with the axe of greatest moment, whose cosines shall be equal to the square root of the expression $\frac{L(M-N)}{M(L-N)}$, and a cone be conceived having those lines as focals, and touching, moreover, the plane of the impressed couple, the entire motion of this body will consist in the rotation of this cone on the invariable plane, with a variable velocity, while the plane revolves round its own axis with an uniform velocity.



Let ACB be the mean section of the ellipsoid. ON, ON′ the cyclic axes; then if the plane of the impressed couple coincides with any of the principal planes, the cones round the cyclic axes as focals, become planes also; and the axis of rotation coincides with one of the axes of the figure.

Again, if the plane of the impressed couple intersects the mean plane between N and C, it will envelope the cone whose focals are ON, ON′, and whose internal axe is therefore OA. But if it intersect between A and N, it will envelope the cone whose focals are ON, OM, and whose internal axe is OC. Whence the range in the former case — which may be taken as the measure of the stability of rotation round the axe whose moment is the

greatest,—is to the range in the latter case—which may also be assumed as the representative of the stability of rotation round that axe whose moment of inertia is the least,—as the supplement of the angle between the cyclic axes of the ellipsoid is to the angle between those axes.

It is also evident, that the sign of the spherical elliptic arc will depend on the sign of the binomial $(b^2 - k^2)$ in (170). The signs of $\varkappa t$ and σ being contrary when $b > k$, they will be the same when $b < k$. We may therefore infer, that the direction in which the angle σ shall be described will depend upon the position of the axis k in the body; whether it lies within the region between the planes of the circular sections of the ellipsoid, or without.

From the theorem established in (IV.) we may infer that the product of the sines of the angles, which the cyclic axes of the body make with the plane of the impressed couple, is constant during the motion; for the cyclic axes of the ellipsoid of moments are the focals of the cone supplemental to the invariable cone.

LXXXIX. To determine the angle between the instantaneous axis of rotation and the line of the nodes.

Let this angle be δ'. The cosines of the angles which the axis of the impressed couple makes with the axes of coordinates being as before l, m, n, let the cosines of the angles which the line of the nodes makes with the same axes be $l,'' m,'' n,''$. λ, μ, ν, are the angles which the instantaneous axis of rotation makes with the same axes.

$$\text{Then } \cos\delta' = l'' \cos\lambda + m'' \cos\mu + n'' \cos\nu \quad . \quad . \quad (a).$$

As the line of the nodes lies in the plane of the impressed couple.

$$0 = l'' l + m'' m + n'' n \quad . \quad . \quad . \quad . \quad (b)$$

and as it is perpendicular to the axis of z,

$$0 = l''\cos\frac{\pi}{2} + m''\cos\frac{\pi}{2} + n''\cos 0, \text{ or } 0 = n''. \quad . \quad (c):$$

and the two former become

$$\cos\delta' = l''\cos\lambda + m''\cos\mu, \quad 0 = l''l + m''m, \text{ and } l''^2 + m''^2 = 1.$$

from the last two equations, $m'' = \dfrac{l}{\sqrt{l^2+m^2}}$, $l'' = \dfrac{-m}{\sqrt{l^2+m^2}}$

whence $\cos\delta = \dfrac{l\cos\mu - m\cos\lambda}{\sqrt{l^2+m^2}}$; now $l = \dfrac{x}{k}$, $m = \dfrac{y}{k}$, $\cos\lambda = \dfrac{\mathrm{P}x}{a^2}$, $\cos\mu = \dfrac{\mathrm{P}y}{b^2}$

$$\text{or } \cos\delta = \frac{\mathrm{P}(a^2-b^2)xy}{a^2 b^2 \sqrt{x^2+y^2}} \quad . \quad . \quad (173).$$

When two of the moments of inertia are equal, L = M suppose, $a = b$, and $\cos\delta' = 0$, or $\delta' = 90$. Whence we infer that when the body is a solid of revolution, the angle between the instantaneous axis of rotation and the line of the nodes is always a right angle.

The angle δ' is also a right angle whenever the axis of the impressed couple lies in one of the planes of the principal sections of the ellipsoid, for then $x = 0$, or $y = 0$.

XC. To determine the angle between the line of the nodes and the axis u of the centrifugal couple.

Let χ be the angle which the axis u of the centrifugal couple makes with a fixed line, ψ the angle which the line of the nodes makes with the same fixed line; then as the line of the nodes and u are in the plane of the impressed couple, see (L.), the angle to be determined is $(\chi - \psi)$.

Now the cosines of the angles which u makes with the axes, are $\dfrac{dx}{ds}$, $\dfrac{dy}{ds}$, $\dfrac{dz}{ds}$; whence $\cos(\chi-\psi) = l''\dfrac{dx}{ds} + m''\dfrac{dy}{ds} + n''\dfrac{dz}{ds}$.

The values of l'', m'', n'' were found in the last article,

$$l'' = \frac{m}{\sqrt{l^2+m^2}}, \; m'' = \frac{-l}{\sqrt{l^2+m^2}}, \; n'' = 0.$$

We may hence deduce $\cos(\chi-\psi) = \dfrac{k}{\sqrt{k^2-z^2}}\left\{\dfrac{y}{k}\dfrac{dx}{ds} - \dfrac{x}{k}\dfrac{dy}{ds}\right\}$

$$\text{but } \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds}, \quad \frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds}$$

$$\text{and } \frac{dx}{dt} = f\frac{(b^2 - c^2)}{b^2 c^2} yz, \quad \frac{dy}{dt} = f\frac{(c^2 - a^2)}{a^2 c^2} xz, \text{ as in (121).}$$

$$\text{Whence } \frac{ds}{dt}\cos(\chi - \psi) = \frac{fz}{\sqrt{k^2 - z^2}}\left\{\frac{a^2b^2y^2 + a^2b^2x^2 - a^2c^2y^2 - b^2c^2x^2}{a^2 b^2 c^2}\right\}$$

The part within the braces is $\frac{(k^2 - c^2)}{c^2}$, and $\frac{ds}{dt} = f \tan\theta$;

$$\text{whence } \cos(\chi - \psi) = \frac{z}{\sqrt{k^2 + z^2}}\left(\frac{k^2 - c^2}{c^2}\right)\cot\theta.$$

ρ being the angle between the axes c and k, $\cos\rho = \frac{z}{k}$. Introducing this value of z into (120) and the trigonometrical functions of α and β the principal semiangles of the invariable cone, as given in (156),

$$\tan\theta = \left(\frac{k^2 - c^2}{c^2}\right)\sqrt{\frac{\cos^2\rho - \cos^2\alpha\cos^2\beta}{\sin^2\alpha\sin^2\beta}} \qquad (174).$$

$$\text{whence } \cos^2(\chi - \psi) = \frac{\sin^2\alpha\sin^2\beta}{\sin^2\rho - \cos^2\alpha\cos^2\beta\tan^2\rho}$$

$$\text{and } \tan^2(\chi - \psi) = \frac{(\sin^2\alpha - \sin^2\rho)(\sin^2\rho - \sin^2\beta)}{\sin^2\alpha\sin^2\beta\cos^2\rho} \qquad (175).$$

This formula leads us to infer that when $\alpha = \beta$, $\chi - \psi$ is always 0, or $\chi = \psi$; whence the axis of the centrifugal couple, when the solid is one of revolution, always coincides with the line of the nodes.

Again, when $\rho = \alpha$, or $\rho = \beta$, $\chi = \psi$; that is, whenever the axis of the impressed couple lies in one of the principal planes of the solid, the axis of the centrifugal couple coincides with the line of the nodes.

XCI. Let segments equal to R measured from the centre be assumed on the three principal axes of the body, the sum of the areas described by the projections of those lines on the plane of the impressed couple is constant.

Let S_c be the area described by the projection of a portion of the axis of c equal to R on the plane of the impressed couple. Then the projection of R on this plane is R $\sin\rho$, and the differential of the area $\frac{dS_c}{dt} = \frac{1}{2}R^2 \sin^2\rho \frac{d\psi}{dt}$; now $\sin^2\rho = \frac{k^2 - z^2}{k^2}$

and $\frac{d\psi}{dt} = \kappa \left\{1 - \left(\frac{k^2 - c^2}{c^2}\right)\left(\frac{z^2}{k^2 - z^2}\right)\right\}$; whence

$$\frac{dS_c}{dt} = \frac{1}{2}\kappa R^2 \left\{1 - \frac{z^2}{c^2}\right\} \text{ (a).}$$

In like manner,

$$\frac{dS_a}{dt} = \frac{1}{2}\kappa R^2 \left\{1 - \frac{x^2}{a^2}\right\}; \qquad \frac{dS_b}{dt} = \frac{1}{2}\kappa R^2 \left\{1 - \frac{y^2}{b^2}\right\}.$$

Whence $\frac{dS_a}{dt} + \frac{dS_b}{dt} + \frac{dS_c}{dt} = \frac{1}{2}R^2\kappa \left\{3 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right\} = R^2\kappa.$

Or $\qquad S_a + S_b + S_c = R^2\kappa t + \text{constant} \qquad . \quad . \quad (176).$

XCII. Should the lengths R instead of being equal, be proportional to the square roots of the moments of inertia round the corresponding axes; the sum of the areas described by the projections of those lines, on the plane of the impressed couple, is still proportional to the time.

Let $R^2 = \frac{N}{W} = \frac{n c^2}{W}$. W being a constant. Then (a) in the last article may be changed into the following, $\frac{dS_c}{dt} = \kappa \frac{n}{W}(c^2 - z^2)$.

Whence $S_a + S_b + S_c = \frac{n}{W}\kappa(a^2 + b^2 + c^2 - k^2)t + \text{con.} \; (177).$

The square of the angular velocity round the instantaneous axis of rotation is always proportional to the area of the diametral section of the ellipsoid perpendicular to this axis.

SECTION V.

XCIII. In the preceding sections formulæ are given which enable us to determine the position of the axis of rotation, and of the axis of the plane of the impressed couple, with reference to fixed lines taken within the body. It still, however, remains to determine the positions not only of those lines, but of the fixed lines within the body, relatively to absolute space. True, we may by transformations of co-ordinates, and by the choice of other variables, obtain solutions from the formulæ already established, by methods which are however tedious, complex, and not a little obscure. It will be found not only the most direct, but by far the most elegant method of procedure, to conduct the investigation independently, and start from first principles.

As the body must now be referred to fixed lines in space, it is no less obvious than natural that we should assume the plane of the impressed couple as one of the co-ordinate planes. Let this plane be taken as that of $x'y'$, its axis the axis of z'. Moreover, let the plane of the greatest and least principal axes of the ellipsoid of moments coincide with the plane of $x'z'$, at the beginning of the time t. The instantaneous axis of rotation will be in the same plane at the same epoch, and will make with the vertical axis k, an angle whose tangent is given by the equation

$$\tan^2 \Theta = \frac{(a^2 - k^2)(k^2 - c^2)}{a^2 c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (178).$$

This may easily be shown, for the perpendicular from the centre on a tangent through the vertex of k, a semidiameter of an ellipse whose semiaxes are a and c, makes with k an angle, whose tangent is given by the last formula.

In like manner, for the principal section whose semiaxes are b and c, we get

$$\tan^2 \Theta' = \frac{(b^2 - k^2)(k^2 - c^2)}{b^2 c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (179).$$

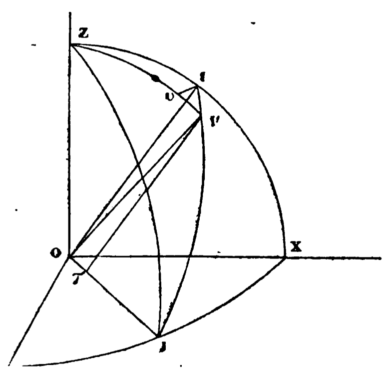
We now proceed to establish the following proposition: —

XCIV. *The area described by the axis u of the centrifugal couple, on the plane of the impressed couple, varies as the time.*

The following relations were established in (130), (116), (113), (114):

$$\frac{\frac{ds}{dt}}{\mathrm{P}'u} = \frac{\omega'}{f}, \quad \frac{ds}{dt} = f \tan\theta, \quad \omega = \frac{f}{\mathrm{P}}, \text{ and } \varkappa = \frac{f}{k};$$

$$\text{whence} \quad \frac{\omega'}{\omega} = \frac{\mathrm{P}k}{\mathrm{P}'u}\,\varkappa \tan\theta \quad \ldots\ldots \quad (180).$$



Let O be the centre of a sphere whose radius is 1, Z the point in which the axis of the plane of the impressed couple meets it, and O I the direction of the instantaneous axis of rotation at the end of the time t. Let the plane which passes through these lines O Z, O I, or the plane of the centrifugal couple coincide with the plane of xz at the same instant. Then the axis of Y will at that instant be the axis of the centrifugal couple; and the perpendicular from the centre on the tangent plane to the ellipsoid, at the point where the axis of Y intersects it, will be the axis of rotation due to the centrifugal couple, See (LXII.) Let the direction of this perpendicular be O J. Through O I J let a plane be drawn. If we assume along O I, O J the instantaneous axes of rotation, lengths O I, Oτ, proportional to the angular velocities ω, ω' round these axes, the diagonal O I′ of the parallelogram constructed with those lines as sides, will represent in direction the instantaneous axis of rotation at the end of the time $t + dt$.

Let O I, O τ taken in this proportion, be the sides of the paral-

lelogram; the diagonal OI' will be the contemporaneous position of this axis of rotation.

Let the angle $ZOI = \theta$, $YOJ = \theta'$, also let δ be the angle between the planes of IOJ and ZOX. Then, as the instantaneous axis of rotation due to the centrifugal couple lies always in the plane of the impressed couple, see (LXI.), the line OJ is in the plane of xy; and the angle $JOX = \frac{\pi}{2} - \theta'$. Let χ be the angle which the vector arc θ makes with a fixed great circle of the sphere passing through Z. The instantaneous axis having moved into the position OI', the arc ZI will have moved into the position ZI', or through the angle $d\chi$, in the time dt. Let $I\upsilon$ be an arc of a great circle perpendicular to ZI', and as $II'\upsilon$ is an infinitesimal right-angled triangle, we shall have $II' \sin\delta = I\upsilon = \frac{d\chi}{dt} \sin\theta$. Again, as IJX is a spherical triangle right angled at X; $\sin IJ : \sin JX :: 1 : \sin\delta$,

$$\text{or } \sin IJ = \frac{\cos\theta'}{\sin\delta}.$$

We are also given by the construction,

$$\frac{\omega'}{\omega} = \frac{\sin II'}{\sin IJ} = \frac{II' \sin\delta}{\cos\theta'} = \frac{d\chi}{dt}\frac{\sin\theta}{\cos\theta'}.$$

and (131) gives $\frac{\omega'}{\omega} = \frac{P\,k}{P'\,u} \times \tan\theta.$

Equating those values of $\frac{\omega'}{\omega}$, and introducing the relations $P = k\cos\theta$, $P' = u\cos\theta'$, we get

$$u^2 \frac{d\chi}{dt} = k^2 \,\varkappa \quad . \; . \; . \; . \; . \; . \; . \; . \quad (181).$$

Now $u^2 \frac{d\chi}{dt}$ is the elementary area described on the plane of the impressed moment, by the semidiameter u of the ellipsoid which coincides with the axis of the centrifugal couple; whence

the area described by this semidiameter is proportional to the time, or

$$\int u^2 \frac{d\chi}{dt}\, dt = k^2 \varkappa\, t + \text{constant} \quad . \quad . \quad . \quad . \quad (182).$$

XCV. To determine the position of the instantaneous axis of rotation in absolute space, at the end of any given time.

If along the axes of rotation due to the impressed and centrifugal couples, we take two lines to represent the angular velocities due to those couples, the diagonal of the parallelogram, constructed with those lines as sides, will represent the instantaneous position of the axis of rotation.

Now if we turn to the fig. at p. 99., we shall see that

$\sin \text{I I}' : \sin \text{I J} :: \omega' : \omega$, and ultimately $\frac{d\sigma}{dt} = \sin \text{I I}'$; whence

$$\frac{d\sigma}{dt} = \frac{\omega'}{\omega} \sin \text{I J}; \qquad \text{or } \left(\frac{d\sigma}{dt}\right)^2 = \frac{\omega'^2}{\omega^2} - \frac{\omega'^2}{\omega^2} \cos^2 \text{I J} \quad . \quad (a).$$

The general formula for the element of an arc measured on the surface of a sphere is

$$\left(\frac{d\sigma}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\chi}{dt}\right)^2. \quad \text{Whence}$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{\omega'^2}{\omega^2} - \sin^2\theta \left(\frac{d\chi}{dt}\right)^2 - \frac{\omega'^2}{\omega^2} \cos^2 \text{I J} \quad . \quad . \quad . \quad (b).$$

We must now reduce this formula.

In (181) it was shown that $\frac{d\chi}{dt} = \frac{\varkappa\, k^2}{u^2}$, and in (131) that $\frac{\omega'}{\omega} = \frac{\varkappa k^2}{\text{P}' u} \sin\theta$. Making the substitutions suggested by those transformations, we get

$$\left(\frac{d\theta}{dt}\right)^2 = k^4\, \varkappa^2\, \sin^2\theta \left\{ \frac{1}{\text{P}'^2\, u^2} - \frac{1}{u^4} \right\} - \frac{\omega'^2}{\omega^2} \cos^2 \text{I J} \quad . \quad . \quad . \quad (c).$$

We shall now proceed to reduce the first term of the second

member of this formula. To facilitate the calculations, let

$$Q = k^4 x^2 \sin^2 \theta \left[\frac{1}{P'^2 u^2} - \frac{1}{u^4} \right].$$

Multiplying by $\left(\frac{ds}{dt}\right)^4$, we shall have

$$\left(\frac{ds}{dt}\right)^4 Q = k^4 x^2 \sin^2 \theta \left[\frac{\left(\frac{ds}{dt}\right)^2}{P'^2 u^2} \left(\frac{ds}{dt}\right)^2 - \left(\frac{\frac{ds}{dt}}{u}\right)^4 \right] \quad . \quad . \quad (d).$$

s, it must be borne in mind, is the arc of the invariable conic, and zyz are the co-ordinates of the vertex of k referred to the principal planes of the ellipsoid.

Now if we turn to (LXIV.) and (LXV.) we shall there find

$$\frac{\left(\frac{ds}{dt}\right)^2}{P'^2 u^2} = \frac{\left(\frac{dx}{dt}\right)^2}{a^4} + \frac{\left(\frac{dy}{dt}\right)^2}{b^4} + \frac{\left(\frac{dz}{dt}\right)^2}{c^4}$$

$$\frac{\left(\frac{ds}{dt}\right)^2}{u^2} = \frac{\left(\frac{dx}{dt}\right)^2}{a^2} + \frac{\left(\frac{dy}{dt}\right)^2}{b^2} + \frac{\left(\frac{dz}{dt}\right)^2}{c^2} \quad . \quad . \quad (e).$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

Making the substitutions suggested by these transformations, we obtain

$$\frac{a^4 b^4 c^4 \left(\frac{ds}{dt}\right)^4 Q}{x^2 k^4 \sin^2 \theta} = \left\{ \begin{array}{l} b^4 c^4 \left(\frac{dx}{dt}\right)^2 \left[\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - 2\frac{a^2}{b^2}\left(\frac{dy}{dt}\right)^2 \right] + \\ a^4 c^4 \left(\frac{dy}{dt}\right)^2 \left[\left(\frac{dz}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 - 2\frac{b^2}{c^2}\left(\frac{dz}{dt}\right)^2 \right] + \\ a^4 b^4 \left(\frac{dz}{dt}\right)^2 \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 - 2\frac{c^2}{a^2}\left(\frac{dx}{dt}\right)^2 \right]. \end{array} \right\} \quad (f).$$

Making the obvious reductions in this equation;

$$\frac{\left(\frac{ds}{dt}\right)^4 Q}{x^2 k^4 \sin^2\theta} = \left(\frac{1}{c^2}-\frac{1}{b^2}\right)^2 \left(\frac{dy}{dt}\right)^2 \left(\frac{dz}{dt}\right)^2 +$$

$$\left(\frac{1}{c^2}-\frac{1}{a^2}\right)^2 \left(\frac{dz}{dt}\right)^2 \left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2 \left(\frac{dx}{dt}\right)^2 \left(\frac{dy}{dt}\right)^2 \quad . \quad . \quad (g).$$

We have also, (121),

$$\left(\frac{dx}{dt}\right)^2 = f^2 \frac{(b^2-c^2)^2}{b^4 c^4} y^2 z^2, \qquad \left(\frac{dy}{dt}\right)^2 = f^2 \frac{(a^2-c^2)^2}{a^4 c^4} x^2 z^2,$$

$$\left(\frac{dz}{dt}\right)^2 = f^2 \frac{(a^2-b^2)^2}{a^4 b^4} x^2 y^2; \text{ or reducing,}$$

$$\left(\frac{1}{c^2}-\frac{1}{b^2}\right)^2 \left(\frac{dy}{dt}\right)^2 \left(\frac{dz}{dt}\right)^2 = \frac{f^4 (b^2-c^2)^2 (a^2-c^2)^2 (a^2-b^2)^2 x^4 y^2 z^2}{a^8 b^8 c^8} \quad . \quad (h).$$

Finding similar values for the other symmetrical expressions, substituting, introducing the relation, $x^2 + y^2 + z^2 = k^2$; and writing for $\frac{ds}{dt}$ its value $f \tan\theta$; we obtain

$$Q = \left[\frac{x\, k^3 \sin\theta\, (a^2-b^2)\,(a^2-c^2)\,(b^2-c^2)\, x\, y\, z}{a^4 b^4 c^4 \tan^2\theta}\right]^2 \quad . \quad . \quad . \quad (j).$$

We have now to compute the term $\frac{\omega'^2}{\omega^2} \cos^2 \text{I J}$.

In (LXVI.) it was shown that the angle between the axes of rotation due to the impressed and centrifugal couples, was given by the formula

$$\cos \text{I J} = \frac{p p' + q q' + r r'}{\omega\, \omega'};$$

$$\text{whence } \frac{\omega'^2}{\omega^2} \cos^2 \text{I J} = \left(\frac{p p' + q q' + r r'}{\omega^2}\right)^2.$$

In (112) and (122) it was shown that

$$p = \frac{fx}{a^2},\ p' = \frac{f^2 (b^2-c^2)\, y\, z}{a^2 b^2 c^2}; \text{ or } p p' = \frac{f^3 x y z}{a^2 b^2 c^2} \left(\frac{b^2-c^2}{a^2}\right).$$

Finding analogous expressions for $q q'$ and $r r'$;

$$p p' + q q' + r r' = \frac{f^3 x y z}{a^2 b^2 c^2} \left\{ \frac{b^2-c^2}{a^2} + \frac{c^2-a^2}{b^2} + \frac{a^2-b^2}{c^2} \right\}.$$

Now $\omega = \frac{f}{\mathrm{P}} = \frac{f}{k \cos\theta}$, as in (114); and

$$\frac{b^2-c^2}{a^2} + \frac{c^2-a^2}{b^2} + \frac{a^2-b^2}{c^2} = \frac{(b^2-c^2)\,(a^2-c^2)\,(a^2-b^2)}{a^2\,b^2\,c^2};$$ whence

$$\frac{\omega'^2}{\omega^2} \cos^2 \mathrm{IJ} = \frac{f^2\,k^4 \cos^4\theta\,(a^2-b^2)^2\,(a^2-c^2)^2\,(b^2-c^2)^2\,x^2 y^2 z^2}{a^8\,b^8\,c^8}. \text{ (k).}$$

Multiplying this expression, numerator, and denominator, by $\tan^4\theta$, writing $\varkappa k$ for f, and in the expression

$$\left(\frac{d\theta}{dt}\right)^2 = \mathrm{Q} - \frac{\omega'^2}{\omega^2} \cos^2 \mathrm{IJ}, \quad . \quad . \quad . \quad \text{(m.)}$$

substituting for the terms of the second member the values found in the preceding equations, reducing and taking the square root,

$$\frac{d\theta}{dt} = \frac{\varkappa k^3 \sin\theta \cos\theta\,(a^2-b^2)(b^2-c^2)(a^2-c^2)\,xyz}{a^4\,b^4\,c^4 \tan^2\theta} \quad . \quad . \quad . \quad (183).$$

XCVI. We have now to express xyz in terms of θ.

Combining the simultaneous equations of the ellipsoid of moments, of the concentric sphere, and of the perpendicular from the centre on the tangent plane to the ellipsoid; namely,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad x^2+y^2+z^2 = k^2, \text{ and}$$

$$\frac{k^2}{\mathrm{P}^2} = 1 + \tan^2\theta = k^2 \left\{ \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right\};$$

we obtain from these equations,

$$\left.\begin{aligned} \frac{x^2}{a^4} &= \frac{[b^2\,c^2 \tan^2\theta - (b^2-k^2)\,(k^2-c^2)]}{k^2\,(a^2-c^2)\,(a^2-b^2)} \\ \frac{y^2}{b^4} &= \frac{[(a^2-k^2)\,(k^2-c^2) - a^2\,c^2 \tan^2\theta]}{k^2\,(a^2-b^2)\,(b^2-c^2)} \\ \frac{z^2}{c^4} &= \frac{[a^2\,b^2 \tan^2\theta + (a^2-k^2)\,(b^2-k^2)]}{k^2\,(a^2-c^2)\,(b^2-c^2)} \end{aligned}\right\} \quad . \quad . \quad (184).$$

Substituting these values of $x\,y\,z$ in (183), the resulting equation will become

$$\frac{dt}{d\theta}=\frac{a^2\,b^2\,c^2\sin\theta\sec^3\theta}{\varkappa\sqrt{[a^2b^2\tan^2\theta+(a^2-k^2)(b^2-k^2)][b^2c^2\tan^2\theta-(b^2-k^2)(k^2-c^2)][(a^2-k^2)(k^2-c^2)-a^2c^2\tan^2\theta]}}\quad(185).$$

This is an elliptic function of the first order, which may be reduced to the usual form by assuming

$$a^2b^2c^2\tan^2\theta=(k^2-c^2)[b^2(a^2-k^2)\cos^2\lambda+a^2(b^2-k^2)\sin^2\lambda]\ (186).$$

Before we proceed further, we shall give the geometrical interpretation of this assumption.

Let a cone be conceived whose internal axe shall coincide with the axis of the plane of the impressed couple, or with the axis of z, and whose principal arcs shall be the greatest and least elongations of the instantaneous axis of rotation from the axis of the impressed couple. This cone will generate on the surface of the sphere a spherical conic, the tangents of whose principal arcs are given as in (178) by the equations,

$$\tan^2\alpha''=\frac{(a^2-k^2)\,(k^2-c^2)}{a^2\,c^2},\ \tan^2\beta''=\frac{(b^2-k^2)\,(k^2-c^2)}{b^2\,c^2}\ ..(187).$$

This cone may be named the *cone of nutation*.

Now, if from the centre of this curve, the vector arc θ is drawn to a point on it; λ is the angle which the perpendicular arc from the centre on the tangent arc through the vertex of θ, makes with the principal arc α''.

To simplify the results, let

$$\begin{aligned}X &= b^2\,c^2\tan^2\theta-(b^2-k^2)\,(k^2-c^2),\\ Y &= (a^2-k^2)\,(k^2-c^2)-a^2\,c^2\tan^2\theta,\\ Z &= a^2\,b^2\tan^2\theta+(a^2-k^2)\,(b^2-k^2);\end{aligned}$$

and the equation (185) will become

$$\frac{dt}{d\theta}=\frac{a^2\,b^2\,c^2\tan^2\theta}{\varkappa\sin\theta\cos\theta\ \sqrt{X.Y.Z.}}\quad .\quad .\quad .\quad (188).$$

If we differentiate (186), and make the transformations

resulting from that assumption, we shall get the following relations:

$$\frac{a^2}{k^2} \mathrm{X} = (a^2-b^2)(k^2-c^2) \cos^2 \lambda;$$

$$\frac{b^2}{k^2} \mathrm{Y} = (a^2-b^2)(k^2-c^2) \sin^2 \lambda; \text{ and}$$

$$\frac{c^2}{k^2} \mathrm{Z} = (a^2-k^2)(b^2-c^2) \cos^2 \lambda + (b^2-k^2)(a^2-c^2) \sin^2 \lambda \quad . . (189).$$

By the help of these transformations, equation (188) takes the form

$$dt = \frac{\mp\, abc}{\varkappa k \sqrt{(a^2-k^2)(b^2-c^2)}} \times \frac{d\lambda}{\sqrt{1-\frac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)} \sin^2 \lambda}} \,.\,(190);$$

which is precisely the same elliptic function we found in (156), differing from it only in the amplitude λ, and the sign. When $b > a$ the positive sign must be taken. We shall show presently that φ and λ have opposite signs.

This formula may be thus written, as in (157.):

$$t = \frac{\mp\, c^2 \sec \alpha}{k^2 \varkappa \sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2 b^2}}} \times \cos \epsilon \int \frac{d\lambda}{\sqrt{1-\sin^2 \epsilon \sin^2 \lambda}} \,.\,(190).$$

When the functions are complete they become identical, as they manifestly should be, because the maximum and minimum values of θ, the greatest and least elongations of the instantaneous axis of rotation from the axis of the plane of the impressed couple, should be given by the same formula, whatever system of axes we choose; as this value must be independent of the position of any axes chosen at will, being a function of the constitution of the body, and of the magnitude and position of the impressed couple.

XCVII. To determine the angle χ, which θ the vector arc, drawn from the vertex of k, makes with a fixed plane passing through k the axis of the impressed couple.

Resuming the equation $\frac{d\chi}{dt} = \frac{\varkappa k^2}{u^2}$, established in (182), we have now to express u^2 in terms of λ.

If we turn to (128), we there find

$$\frac{\left(\frac{ds}{dt}\right)^2}{u^2} = \frac{\left(\frac{dx}{dt}\right)^2}{a^2} + \frac{\left(\frac{dy}{dt}\right)^2}{b^2} + \frac{\left(\frac{dz}{dt}\right)^2}{c^2}, \text{ or}$$

$$\frac{a^2b^2c^2}{u^2}\left(\frac{ds}{dt}\right)^2 = b^2\,c^2\left(\frac{dx}{dt}\right)^2 + a^2\,c^2\left(\frac{dy}{dt}\right)^2 + a^2\,b^2\left(\frac{dz}{dt}\right)^2.$$

Eliminating $\frac{dz}{dt}$ by the relation, $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$

$$\frac{a^2\,b^2\,c^2}{u^2}\left(\frac{ds}{dt}\right)^2 = a^2\,b^2\left(\frac{ds}{dt}\right)^2 - b^2(a^2-c^2)\left(\frac{dx}{dt}\right)^2 - a^2\,(b^2-c^2)\left(\frac{dy}{dt}\right)^2;$$

now $\left(\frac{dx}{dt}\right)^2 = \frac{f^2\,(b^2-c^2)^2\,z^2y^2}{b^4\,c^4}$, $\left(\frac{dy}{dt}\right)^2 = \frac{f^2\,(a^2-c^2)^2\,x^2z^2}{a^4\,c^4}$.

Making these substitutions, we shall find, $\frac{a^2b^2c^2}{u^2}\left(\frac{ds}{dt}\right)^2 =$

$$a^2\,b^2\left(\frac{ds}{dt}\right)^2 - \frac{(a^2-c^2)(b^2-c^2)\,z^2}{a^2\,b^2\,c^4}f^2(a^2b^2y^2 + a^2b^2x^2 - a^2c^2y^2 - b^2c^2x^2).$$

Eliminating x^2 and y^2 by the equations of the ellipsoid and sphere; introducing also, the relations $\frac{ds}{dt} = f\tan\theta$, and

$$a^2\,b^2\,c^4\tan^2\theta = (a^2-c^2)\,(b^2-c^2)\,k^2\,z^2 - c^4\,(a^2-k^2)\,(b^2-k^2),$$

given in (120), we get

$$\frac{k^2}{u^2} = \frac{a^2\,b^2\,c^2\tan^2\theta + (a^2-k^2)\,(b^2-k^2)\,(c^2-k^2)}{a^2\,b^2\,c^2\tan^2\theta} \quad . \; . \; . \; (191).$$

In this equation substituting the value of $\tan\theta$, given in terms of λ in (186), we obtain

$$\frac{1}{u^2} = \frac{(a^2-k^2)\cos^2\lambda + (b^2-k^2)\sin^2\lambda}{b^2(a^2-k^2)\cos^2\lambda + a^2\,(b^2-k^2)\sin^2\lambda} \quad . \; . \; (192).$$

Now this may easily be reduced to the form

$$\frac{1}{u^2} = \frac{\frac{1}{b^2} - \frac{(a^2-b^2)}{b^2(a^2-k^2)} \sin^2 \lambda}{1 - \frac{k^2}{b^2}\left(\frac{a^2-b^2}{a^2-k^2}\right) \sin^2 \lambda}, \quad . \quad (192^*).$$

but it has been already shown that

$\frac{k^2}{b^2}\frac{(a^2-b^2)}{(a^2-k^2)} = e^2$; e being the eccentricity of the plane elliptic base of the invariable cone in (169); whence

$$\frac{k^2}{u^2} = 1 - \left(\frac{b^2-k^2}{b^2}\right)\left[\frac{1}{1 - e^2 \sin^2 \lambda}\right] \quad . \quad . \quad (193).$$

Introducing this value of $\frac{k^2}{u^2}$ into the equation $\chi = \kappa \int \frac{k^2}{u^2} dt$, writing for dt its value given in (190), and integrating, we obtain the final result,

$$\chi = \kappa t \pm \frac{ac}{bk} \frac{(b^2-k^2)}{\sqrt{(a^2-k^2)(b^2-c^2)}} \int \frac{d\lambda}{[1-e^2 \sin^2 \lambda]\sqrt{1-\sin^2 \epsilon \sin^2 \lambda}} \quad (194).$$

The negative sign to be taken when $b > k$.

This formula differs from (171) only in the amplitude.

When the functions (171) and (194) are complete, the values of ψ and χ become identical, as they manifestly ought to be, because in (XC.) it was shown that the line of the nodes coincides with the axis of the centrifugal couple whenever the instantaneous axis of rotation lies in one of the principal planes of the ellipsoid.

XCVIII. If we eliminate $\tan \theta$ between (158) and (186), we get the following relation between ϕ and λ:

$$\tan \phi \tan \lambda = \sec \epsilon ; \quad . \quad . \quad . \quad (195);$$

hence $\frac{d\phi}{d\lambda} = -\frac{\sin 2\phi}{\sin 2\lambda}$; or ϕ and λ have opposite signs.

But these angles differ in their origin by a right angle, since ϕ

is measured from the plane of bc, while λ is measured from that of ac; adding $\frac{\pi}{2}$ to λ to make their origins coincide,

$$\tan \varphi \cot \lambda = \sec \varepsilon.$$

Now when the ellipsoid is a figure of revolution, a equal to b suppose, the invariable cone becomes a right cone of revolution, whence the angles between its focals vanish, or $\varepsilon = 0$. Therefore φ is always equal to λ; that is, the amplitudes of the functions are identical throughout their whole extent, as plainly they ought to be, because in this case the line of the nodes always coincides with the axis of the centrifugal couple.

$$\text{when } \varphi = 0,\ \lambda = 0, \text{ and when } \varphi = \frac{\pi}{2},\ \lambda = \frac{\pi}{2}.$$

We may repeat here what has been said in (LXXXVII.), that the expression

$$\frac{a^2 c^2 (b^2 - k^2)}{b^2 k^2 \sqrt{(a^2 - k^2)(b^2 - c^2)}} \int \frac{d\lambda}{[1 - e^2 \sin^2 \lambda] \sqrt{1 - \sin^2 \varepsilon \sin^2 \lambda}}$$

may be transformed into this other,

$$\frac{\tan \beta}{\tan \alpha} \cos \alpha \int \frac{d\lambda}{[1 - e^2 \sin^2 \lambda] \sqrt{1 - \sin^2 \varepsilon \sin^2 \lambda}},$$

which represents, as was shown in the same section, an arc of the spherical conic, supplemental to the invariable spherical ellipse.

The relation between χ and λ is given by the following equation,

$$\begin{aligned} \chi = \frac{\tan \beta}{\tan \alpha} \cos \alpha \int \frac{d\lambda}{[1 - e^2 \sin^2 \lambda] \sqrt{1 - \sin^2 \varepsilon \sin^2 \lambda}} \\ - \frac{c^2}{k^2} \frac{\sec \alpha}{e' e''} \cos \varepsilon \int \frac{d\lambda}{\sqrt{1 - \sin^2 \varepsilon \sin^2 \lambda}} \end{aligned} \quad . \ (196);$$

in which e′ and e″ are the eccentricities of the principal sections of the ellipsoid passing through the axis c.

XCIX. We may now determine the angular velocity, round the instantaneous axis of rotation, and the nutation of this axis in formulæ of great simplicity.

Since in (190) the time is given in terms of λ, we may reverse the formula and obtain λ a function of t'. (See note, p. 86.) t' in this equation is no longer the same numerical quantity as t in (LXXXIII.); for while all the constants in (155) and (190) are the same, the amplitudes ϕ and λ are different. Accordingly let

$$j : i :: t' : t. \quad \text{or} \frac{j\,t}{t'} = i.$$

Hence $$j\,t = i\,t' \qquad \text{. . . . . (196*).}$$

Let $\lambda = (i\,t')$, or $\lambda = (j\,t)$.

Then in (186) writing for $\tan^2\theta$ its value $\dfrac{k^2}{P^2} - 1$, we get

$$\frac{1}{P^2} = \frac{(a^2 + c^2 - k^2)}{a^2\,c^2}\cos^2\lambda + \frac{(b^2 + c^2 - k^2)}{b^2\,c^2}\sin^2\lambda \quad . \quad \text{(a).}$$

Let $P_{,}$ and $P_{,,}$ be the greatest and least values of P, then

$$\frac{1}{P^2} = \frac{\cos^2\lambda}{P_{,,}^{\,2}} + \frac{\sin^2\lambda}{P_{,}^{\,2}} \quad . \quad . \quad . \quad . \quad \text{(b);}$$

or P is a semidiameter of a plane ellipse whose principal semiaxes are $P_{,}$ and $P_{,,}$.

If Ω and Ω' are put for the greatest and least angular velocities,

$\Omega = \dfrac{f}{P_{,,}}$, $\Omega' = \dfrac{f}{P_{,}}$; we hence get for the angular velocity the very simple expression,

$$\omega^2 = \Omega^2\cos^2(j\,t) + \Omega'^2\sin^2(j\,t) \quad . \quad . \quad (197);$$

or the angular velocity varies as the perpendicular on a tangent to a plane ellipse whose principal semiaxes are Ω and Ω'.

In the same way writing Θ and Θ' for the greatest and least values of θ, the nutation of the instantaneous axis of rotation from the axis of the plane of the impressed couple, we obtain

$$\tan^2\theta = \tan^2\Theta\cos^2(jt) + \tan^2\Theta'\sin^2(jt) \quad . \quad . \ (198).$$

This formula may easily be obtained. Multiplying (b) by k^2, subtracting 1 from the first number, and $\cos^2\lambda + \sin^2\lambda$ from the second, we obtain (198).

C. If now it were possible to eliminate λ from the equations (186) and (196), we should have a direct equation between θ and χ, the polar spherical coordinates of the curve. We cannot do this, but still we may perceive, that as the equations involve the angle χ simply and no trigonometrical function of it, while θ is a periodic function involving sines and cosines of arcs which increase with the time, the curve must be some sort of spiral described on the surface of the sphere. But although this direct elimination is in the general case extremely difficult, perhaps impossible to effect, we may however, be enabled successfully to investigate some of the more important properties of this spiral in the general case, and to give its polar equation in a particular case of rotatory motion.

CI. This spiral, analogous to the herpoloid of Poinsot, has two asymptotic circles on the surface of the sphere.

The angle which the vector arc θ, of a spherical curve, drawn from the origin to any point on the curve, makes with a tangent at that point, is given by the equation

$$\tan\iota = \sin\theta\,\frac{d\chi}{dt} \quad . \quad . \quad . \quad . \ (199).$$

This is evident, because the sides of the elementary right-angled triangle on the surface of the sphere are the element of the arc, the differential of the vector arc θ, and the distance $\sin\theta\frac{d\chi}{dt}$ at that point between two consecutive meridians.

We may transform this equation into

$$\tan\iota = \sin\theta\,\frac{d\chi}{dt}\cdot\frac{dt}{d\theta}.$$

Now in (181) it was shown that $\frac{d\chi}{dt} = \varkappa\frac{k^2}{u^2}$.

$$\text{and } \frac{k^2}{u^2} = \frac{a^2 b^2 c^2 \tan^2\theta + (a^2 - k^2)(b^2 - k^2)(c^2 - k^2)}{a^2 b^2 c^2 \tan^2\theta},$$

(185) gives $\dfrac{dt}{d\theta} = \dfrac{a^2 b^2 c^2 \tan^2\theta}{\varkappa \sin\theta \cos\theta \sqrt{\mathrm{XYZ}}}$; whence

$$\tan\iota = \frac{a^2 b^2 c^2 \tan^2\theta + (a^2 - k^2)(b^2 - k^2)(c^2 - k^2)}{\cos\theta \sqrt{\mathrm{XYZ}}} \quad . \quad (200).$$

Now if we turn to (XCVI.), we shall there find, whatever supposition we make with respect to the magnitude of k, that some one of the factors X, Y, Z, must be essentially positive, and cannot become cypher. In this case, Z is essentially positive. Making $\mathrm{X} = 0$, and $\mathrm{Y} = 0$, successively, we get

$$\tan^2\theta = \frac{(b^2 - k^2)(k^2 - c^2)}{b^2 c^2}, \tan^2\theta = \frac{(a^2 - k^2)(k^2 - c^2)}{a^2 c^2};$$

but when $\mathrm{X} = 0$, or $\mathrm{Y} = 0$, $\tan\iota = \infty$, or ι is a right angle; hence, when θ has either of those values, the spiral touches one or other of the circles whose spherical radii are the values of $\tan\theta$ given above.

If we make θ greater or less than the limiting values just given, either X or Y will become negative, and the value of $\tan\theta$ therefore imaginary. We may hence infer that the spiral on the surface of the sphere is confined between two planes parallel to the plane of the impressed couple; and that it always undulates between two parallel lesser circles of the sphere, having its apsides alternately upon them.

Let $\mathrm{P}_{/}$ and $\mathrm{P}_{//}$ be the greatest and least values of P, the perpendicular from the centre of the ellipsoid of moments on the tangent plane. The area of the spherical belt or zone, within which the undulations of the spiral are contained, is equal to $2\pi k(\mathrm{P}_{/} - \mathrm{P}_{//})$.

CII. It was shown in (LXVIII.) that the instantaneous axis of rotation referred to the principal axes of the body generates a cone of the second degree. We shall now proceed to establish the following remarkable theorem.

The length of the spiral between any two successive apsides is

constant, and equal to a quadrant of the spherical ellipse generated by the cone of rotation.

Let σ be the arc of this spiral,

$$\text{then } \left(\frac{d\sigma}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\chi}{dt}\right)^2$$

(188) (181) and (191) give us $\left(\frac{d\theta}{dt}\right)^2 = \frac{\kappa^2 \sin^2\theta \cos^2\theta \, (\mathrm{X.Y.Z})}{a^4 b^4 c^4 \tan^4\theta}$

$$\left(\frac{d\chi}{dt}\right)^2 \sin^2\theta = \frac{\kappa^2 \sin^2\theta \left[a^2b^2c^2 \tan^2\theta + (a^2-k^2)(b^2-k^2)(c^2-k^2)\right]}{a^4 b^4 c^4 \tan^4\theta}$$

making the requisite substitutions in the general formula for the spherical arc, we find

$$\left(\frac{d\sigma}{dt}\right)^2 = \frac{\kappa^2 \sin^2\theta \cos^2\theta \, (\mathrm{X.Y.Z}) + \kappa^2 \sin^2\theta \left[a^2b^2c^2 \tan^2\theta + (a^2-k^2)(b^2-k^2)(c^2-k^2)\right]^2}{a^4 b^4 c^4 \tan^4\theta} \quad \text{(a).}$$

In (XCVI.) we found

$$\mathrm{X} = b^2 c^2 \tan^2\theta - (b^2-k^2)(k^2-c^2),$$
$$\mathrm{Y} = (a^2-k^2)\,(k^2-c^2) - a^2 c^2 \tan^2\theta,$$
$$\mathrm{Z} = a^2 b^2 \tan^2\theta + (a^2-k^2)\,(b^2-k^2).$$

Substituting these values of X, Y, Z in the preceding formula, squaring the second member, and adding, we shall find, after some rather complicated reductions,

$$\frac{a^2b^2c^2}{\kappa^2 k^4}\left(\frac{d\sigma}{dt}\right)^2 = (a^2+b^2+c^2-2k^2)\sin^2\theta\cos^2\theta + (a^2+b^2+c^2-k^2)\left[\frac{(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2\,b^2\,c^2}\right]\cos^4\theta \quad \text{(b).}$$

We must now reduce this formula to a form suited for integration. In (186) we made the assumption,
$a^2 b^2 c^2 \tan^2\theta = (k^2-c^2)\,\{b^2\,(a^2-k^2)\cos^2\lambda + a^2\,(b^2-k^2)\sin^2\lambda\}$.

Let us continue this assumption: reducing we find

$$\sin^2\theta = \frac{(k^2-c^2)}{k^2}\left\{\frac{b^2\,(a^2-k^2)\cos^2\lambda + a^2\,(b^2-k^2)\sin^2\lambda}{b^2\,(a^2+c^2-k^2)\cos^2\lambda + a^2\,(b^2+c^2-k^2)\sin^2\lambda}\right\} \quad . \quad . \quad \text{(c).}$$

and

$$\cos^2\theta = \frac{a^2\,b^2\,c^2}{k^2\left[b^2(a^2+c^2-k^2)\cos^2\lambda + a^2\,(b^2+c^2-k^2)\sin^2\lambda\right]} \quad . \quad . \quad \text{(d).}$$

Substituting and reducing

$$\frac{\left(\frac{d\sigma}{dt}\right)^2}{\kappa^2k^2(k^2-c^2)} = \frac{(a^2-k^2)\,(a^2+c^2-k^2)\cos^2\lambda + (b^2-k^2)\,(b^2+c^2-k^2)\sin^2\lambda}{\left[b^2(a^2+c^2-k^2)\cos^2\lambda + a^2(b^2+c^2-k^2)\sin^2\lambda\right]^2} \quad . \quad \text{(e).}$$

$\frac{d\sigma}{dt}$ denotes the velocity of the pole of the instantaneous axis of rotation along the spiral which it describes. We thus have the velocity of this point given in terms of λ. We shall return to this expression.

To change the independent variable from t to λ.

Multiply the last equation by the equivalent expressions given in (190), namely

$$\left(\frac{dt}{d\lambda}\right)^2 = \frac{a^2 b^2 c^2}{x^2 k^2 \left[(a^2-k^2)(b^2-c^2)\cos^2\lambda + (b^2-k^2)(a^2-c^2)\sin^2\lambda\right]}$$

and we shall have

$$\frac{\left(\frac{d\sigma}{d\lambda}\right)^2}{k^2-c^2} = \frac{a^2 b^2 c^2 \left[(a^2-k^2)(a^2+c^2-k^2)\cos^2\lambda + (b^2-k^2)(b^2+c^2-k^2)\sin^2\lambda\right]}{\left[b^2(a^2+c^2-k^2)\cos^2\lambda + a^2(b^2+c^2-k^2)\sin^2\lambda\right]^2\left[(a^2-k^2)(b^2-c^2)\cos^2\lambda + (b^2-k^2)(a^2-c^2)\sin^2\lambda\right]} \quad (201).$$

We now proceed to show that this expression may be reduced to an elliptic function of the third order and circular form. To simplify the calculations, write

$$\left.\begin{array}{lll} A=(a^2-k^2)(a^2+c^2-k^2), & C=b^2(a^2+c^2-k^2), & U=(a^2-k^2)(b^2-c^2) \\ B=(b^2-k^2)(b^2+c^2-k^2), & D=a^2(b^2+c^2-k^2), & V=(b^2-k^2)(a^2-c^2) \end{array}\right\} \quad . \quad (202).$$

Making these substitutions, dividing by $a^2 b^2 c^2$, and taking the square root, we obtain

$$\frac{\frac{d\sigma}{d\lambda}}{abc\sqrt{k^2-c^2}} = \left\{\frac{A\cos^2\lambda + B\sin^2\lambda}{C\cos^2\lambda + D\sin^2\lambda}\right\}\frac{1}{\sqrt{(A\cos^2\lambda + B\sin^2\lambda)(U\cos^2\lambda + V\sin^2\lambda)}} \quad (203).$$

To integrate this equation, assume $V\tan^2\lambda = U\tan^2\Phi$ (204).

Introducing the changes arising from this transformation, the last equation may be reduced to

$$\frac{d\sigma}{abc\sqrt{k^2-c^2}} = \frac{U(AD-CB)}{C(DU-CV)\sqrt{AV}}\left[\frac{d\Phi}{\left\{1+\left(\frac{DU-CV}{CV}\right)\sin^2\Phi\right\}\sqrt{1-\left(\frac{AV-BU}{AV}\right)\sin^2\Phi}}\right]$$

$$-\frac{(AV-BU)}{(DU-CV)\sqrt{AV}}\left[\frac{d\Phi}{\sqrt{1-\left(\frac{AV-BU}{AV}\right)\sin^2\Phi}}\right] \quad . \quad . \quad . \quad . \quad (205.)$$

We have now to compute the value of the coefficients, modulus, and parameter of this expression.

From the relations established in (202), we get, writing I and J, for the first and second coefficients,

$$\left.\begin{aligned}
&\mathrm{I} = \frac{\mathrm{U(AD-CB)}\,a\,b\,c\sqrt{k^2-c^2}}{\mathrm{C(DU-CV)}\sqrt{\mathrm{AV}}} = \frac{a\,(b^2-c^2)(b^2+c^2-k^2)\sqrt{a^2-k^2}}{b\,c\sqrt{(b^2-k^2)(k^2-c^2)(a^2-c^2)(a^2+c^2-k^2)}}\\
&\mathrm{J} = \frac{\mathrm{(AV-BU)}\,a\,b\,c\sqrt{k^2-c^2}}{\mathrm{(DU-CV)}\sqrt{\mathrm{AV}}} = \frac{a\,b}{c}\sqrt{\frac{(a^2-k^2)\,(b^2-k^2)}{(k^2-c^2)(a^2-c^2)\,(a^2+c^2-k^2)}}\\
&\text{The parameter} = \frac{\mathrm{DU-CV}}{\mathrm{CV}} = \frac{c^2(a^2-b^2)(k^2-c^2)\,(a^2+b^2-k^2)}{b^2\,(a^2-c^2)(b^2-k^2)\,(a^2+c^2-k^2)}\\
&\text{The square of the modulus} = \frac{\mathrm{AV-BU}}{\mathrm{AV}} = \frac{(a^2-b^2)\,(a^2+b^2-k^2)}{(a^2-c^2)\,(a^2+c^2-k^2)}
\end{aligned}\right\} \quad (206).$$

Let us now take the cone described by the instantaneous axis of rotation, with reference to the principal axes of the body. The equation is given in (LXVIII.), namely,

$$a^2\,(a^2-k^2)\,x^2 + b^2\,(b^2-k^2)\,y^2 + c^2\,(c^2-k^2)\,z^2 = 0\,;$$

and we shall find, writing as before α' and β' for the principal arcs of the spherical ellipse, the intersection of this cone with a concentric sphere, that

$$\left.\begin{aligned}
&\tan^2\alpha' = \frac{c^2(k^2-c^2)}{b^2(b^2-k^2)},\ \cos^2\alpha' = \frac{b^2\,(b^2-k^2)}{(b^2-c^2)(b^2+c^2-k^2)},\ \sin^2\alpha' = \frac{c^2\,(k^2-c^2)}{(b^2-c^2)(b^2+c^2-k^2)}\\
&\tan^2\beta' = \frac{c^2(k^2-c^2)}{a^2(a^2-k^2)},\ \cos^2\beta' = \frac{a^2\,(a^2-k^2)}{(a^2-c^2)(a^2+c^2-k^2)},\ \sin^2\beta' = \frac{c^2\,(k^2-c^2)}{(a^2-c^2)(a^2+c^2-k^2)}
\end{aligned}\right\} \quad (207).$$

If we write $2\,\varepsilon'$ for the angle between the focals of this cone, we know from (XVIII.) that its value, in terms of the principal arcs of the spherical ellipse, is given by the equation

$$\tan^2\varepsilon' = \frac{\cos^2\beta' - \cos^2\alpha'}{\cos^2\alpha'}.$$

Substituting the particular values of these functions just given, we obtain

$$\tan^2 \varepsilon' = \frac{c^2 (a^2 - b^2)(k^2 - c^2)(a^2 + b^2 - k^2)}{b^2 (a^2 - c^2)(b^2 - k^2)(a^2 + c^2 - k^2)}.$$

Hence $\tan^2 \varepsilon'$ is the parameter. See (XVIII.)

Let $2\,\eta'$ be the angle between the circular sections of the same cone. It was found in (21) that $\sin^2 \eta' = \dfrac{\sin^2 \alpha' - \sin^2 \beta'}{\sin^2 \alpha'}$,

$$\text{whence } \sin^2 \eta' = \frac{(a^2 - b^2)(a^2 + b^2 - k^2)}{(a^2 - c^2)(a^2 + c^2 - k^2)}$$

or $\sin \eta'$ is the modulus.

Let us compute the value of the first coefficient I.

Making the necessary substitutions, we obtain the resulting expressions:

$$\mathrm{I} = \frac{a(b^2 - c^2)(b^2 + c^2 - k^2)\sqrt{a^2 - k^2}}{b\,c\sqrt{(a^2 - c^2)(b^2 - k^2)(k^2 - c^2)(a^2 + c^2 - k^2)}} = \frac{\cos \beta'}{\cos \alpha' \sin \alpha'}.$$

In like manner we find for the second coefficient J,

$$\mathrm{J} = \frac{a\,b}{c}\sqrt{\frac{(a^2 - k^2)(b^2 - k^2)}{(a^2 - c^2)(k^2 - c^2)(a^2 + c^2 - k^2)}} = \frac{\cos \alpha' \cos \beta'}{\sin \alpha'}.$$

Making all the substitutions just indicated, (205) may be transformed into

$$\text{Arc of spiral} = \frac{\cos \beta'}{\cos \alpha' \sin \alpha'} \int \frac{d\Phi}{[1 + \tan^2 \varepsilon' \sin^2 \Phi]\sqrt{1 - \sin^2 \eta' \sin^2 \Phi}}$$

$$- \frac{\cos \alpha' \cos \beta'}{\sin \alpha'} \int \frac{d\Phi}{\sqrt{1 - \sin^2 \eta' \sin^2 \Phi}} \quad . \quad . \quad (208).$$

It was shown in (XXII.) that if there are two circular elliptic functions of the third order with positive and negative parameters, having the same modulus and amplitude, the parameters being respectively the square of the tangent of the semi-focal angle, and the square of the eccentricity of the plane elliptic base of the cone, the expressions are connected by the following equation, given in (44):

$$\frac{\cos\beta}{\cos\alpha\sin\alpha}\int\frac{d\phi}{[1+\tan^2\epsilon\sin^2\phi]\sqrt{1-\sin^2\eta\sin^2\phi}}-\frac{\cos\alpha\cos\beta}{\sin\alpha}\int\frac{d\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}$$

$$=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-\sin^2\eta\sin^2\phi}}+\tan^{-1}\left\{\frac{e\tan\epsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}\right\}.$$

If now we introduce this relation into the preceding equation (208), we shall obtain for the final result,

$$\text{Arc of spiral}=\frac{\tan\beta'}{\tan\alpha'}\sin\beta'\int\frac{d\Phi}{[1-e'^2\sin^2\Phi]\sqrt{1-\sin^2\eta'\sin^2\Phi}}$$

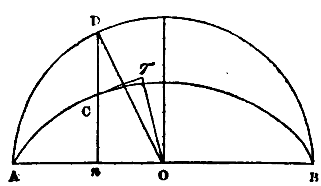
$$+\tan^{-1}\left\{\frac{e'\tan\epsilon'\sin\Phi\cos\Phi}{\sqrt{1-\sin^2\eta'\sin^2\Phi}}\right\}\quad .\quad .\quad .\quad (209).$$

In (IX.) it was established that the expression

$\dfrac{\tan\beta}{\tan\alpha}\sin\beta\displaystyle\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-\sin^2\eta\sin\varphi}}$ is the value

of an arc of the spherical ellipse, the principal angles of whose generating cone are 2α and 2β, the angle between whose circular sections is 2η, and the eccentricity of whose plane elliptic base is e. It was shown in (50) that

$\tan^{-1}\left\{\dfrac{e\tan\epsilon\sin\varphi\cos\varphi}{\sqrt{1-\sin^2\eta\sin^2\varphi}}\right\}$ is the arc of a great circle touching the spherical conic, intercepted between the point of contact and the foot of the perpendicular arc from the centre on the tangent arc.



Make the angle A O D = ϕ, draw the arc D n a secondary to A B, and through C draw the tangent arc C T.

The length of the spiral = elliptic arc A C + circular arc C T.

The length of the spiral between any two successive apsides is found by taking Φ between the limits 0 and $\frac{\pi}{2}$. At those limits

τ vanishes, and the expression becomes the length of a quadrant of the ellipse; hence we obtain this remarkable proposition.

The length of the spiral, between any two of its successive apsides, is equal to a quadrant of the spherical ellipse, described by the pole of the instantaneous axis of rotation, on a concentric sphere which moves with the body.

If we turn to the relation assumed in (204) between λ and Φ for the purpose of facilitating the integrations, and substitute for U and V their values in the equation,

$$\mathrm{V} \tan^2 \lambda = \mathrm{U} \tan^2 \Phi$$

we find $\tan^2 \Phi = \frac{\mathrm{V}}{\mathrm{U}} \tan^2 \lambda$, or $\tan^2 \Phi = \frac{(b^2 - k^2)(a^2 - c^2)}{(a^2 - k^2)(b^2 - c^2)} \tan^2 \lambda.$,

or $\tan \Phi = \cos \epsilon \tan \lambda$.

But λ and the amplitude ϕ used in the investigations in this and the foregoing section, are connected by the relation established in (195),

$$\tan \phi \tan \lambda = \sec \epsilon.$$

Hence ϕ, λ and Φ, the amplitudes used in the preceding investigations, are connected by the equations,

$$\tan \Phi = \cos \epsilon \tan \lambda, \qquad \tan \phi \tan \lambda = \sec \epsilon.$$

Multiplying these equations together, we find that

$$\tan \phi \tan \Phi = 1, \text{ or } \phi + \Phi = \frac{\pi}{2} \quad . \quad . \quad . \quad (210).$$

CIII. Let ϵ, ϵ', ϵ'' be the semifocal angles of the *invariable cone*, of the *cone of rotation*, and of the *cone of nutation* respectively. Then

$$\cos^2 \epsilon = \frac{\cos^2 \alpha}{\cos^2 \beta} = \frac{(b^2 - k^2)(a^2 - c^2)}{(a^2 - k^2)(b^2 - c^2)} \quad . \quad \text{as in (LXXXI);}$$

$$\cos^2 \epsilon' = \frac{\cos^2 \alpha'}{\cos^2 \beta'} = \frac{b^2(b^2 - k^2)(a^2 - c^2)(a^2 + c^2 - k^2)}{a^2(a^2 - k^2)(b^2 - c^2)(b^2 + c^2 - k^2)} \text{ as in (207);}$$

$$\cos^2 \epsilon'' = \frac{\cos^2 \alpha''}{\cos^2 \beta''} = \frac{a^2(b^2 + c^2 - k^2)}{b^2(a^2 + c^2 - k^2)} \text{ from (187)}$$

Whence $\cos \epsilon = \cos \epsilon' \cos \epsilon''$ . . . . . (211).

Let e'' be the eccentricity of the plane base of the cone of nutation.

$$\text{then } e''^2 = \frac{\tan^2\alpha'' - \tan^2\beta''}{\tan^2\alpha''} = \frac{k^2(a^2-b^2)}{b^2(a^2-k^2)}.$$

But it was shown in (169), that $e^2 = \dfrac{k^2(a^2-b^2)}{b^2(a^2-k^2)}$; whence $e = e''$;

or the plane elliptic base of the cone of nutation is similar to that of the invariable cone.

CIV. When the revolving body is very nearly a sphere, as in the case of the planetary bodies, a, b, c are very nearly equal. In this case, the ellipse of rotation is indefinitely greater than the ellipse of nutation, as may thus be shown.

$$\tan^2\alpha' = \frac{c^2(k^2-c^2)}{b^2(b^2-k^2)}, \qquad \tan^2\beta' = \frac{c^2(k^2-c^2)}{a^2(a^2-k^2)}$$

$$\tan^2\alpha'' = \frac{(a^2-k^2)(k^2-c^2)}{a^2c^2}, \ \tan\beta'' = \frac{(b^2-k^2)(k^2-c^2)}{b^2c^2}; \text{ whence}$$

$$\frac{\tan\alpha''}{\tan\alpha'} = \frac{b^2}{c^2}\sqrt{\frac{(a^2-k^2)(b^2-k^2)}{a^2b^2}}, \ \frac{\tan\beta''}{\tan\beta'} = \frac{a^2}{c^2}\sqrt{\frac{(a^2-k^2)(b^2-k^2)}{a^2b^2}} \quad (212).$$

Now when a, b, c are very nearly equal, k also must nearly be equal to one of those quantities; whence as k approaches in magnitude to one of the axes, the above ratio becomes indefinitely small.

As the length of one undulation of the spiral depends solely on the magnitude of the principal arcs of the ellipse of rotation, and is independent of that of nutation; it is evident that when the body approaches in shape to a sphere, several revolutions of the body must occur between one extreme position of the axis of rotation and the one immediately following.

When the body is very nearly a sphere, we may approximate to this number. In this case the ellipses are very nearly circles, and the number of revolutions n will be the ratio of their circumferences; or

$$n = \frac{\text{circumference of circle of rotation}}{\text{circumference of circle of nutation}} = \frac{\sin\alpha'}{\sin\alpha''} = \frac{\tan\alpha'}{\tan\alpha''} = \frac{\mathrm{N}}{\mathrm{L}-\mathrm{N}};$$

or, in the usual notation, $n = \dfrac{\mathrm{C}}{\mathrm{A}-\mathrm{C}}$ nearly, since $a=b=k=c$ nearly.

CV. On the velocity of the pole of the instantaneous axis of rotation along the spiral.

The velocity v along the spiral is the value of the expression $\dfrac{d\sigma}{dt}$. This value has been found, ((e) of CII.) to be in terms of λ,

$$\mathrm{v}^2 = \frac{\kappa^2 k^2(k^2-c^2)\left[(a^2-k^2)(a^2+c^2-k^2)\cos^2\lambda + (b^2-k^2)(b^2+c^2-k^2)\sin^2\lambda\right]}{\left[b^2(a^2+c^2-k^2)\cos^2\lambda + a^2(b^2+c^2-k^2)\sin^2\lambda\right]^2} \quad (213).$$

We shall now proceed to find the maximum and minimum values of v by the ordinary process of differentiation. For this purpose differentiating equation (b) of (CII.) and putting the differential of $\left(\dfrac{d\sigma}{dt}\right)$ equal to 0, we shall obtain

$$0 = \frac{d\theta}{dt}.\ \sin\theta\cos\theta\left\{\mathrm{Q}(\sin^2\theta - \cos^2\theta) - 2\,\mathrm{T}\cos^2\theta\right\} \quad . \quad . \quad (214);$$

writing Q for $a^2+b^2+c^2-2k^2$,

and T for $(a^2+b^2+c^2-k^2)\left[\dfrac{(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2\,b^2\,c^2}\right]$.

In this equation there are four factors, any of which equated to cypher would satisfy the equation; either $\dfrac{d\theta}{dt}=0$, $\sin\theta=0$, $\cos\theta=0$, or $\mathrm{Q}(\sin^2\theta-\cos^2\theta)-2\,\mathrm{T}\cos^2\theta=0$.

We shall now proceed to show that they are all inadmissible except the first.

We cannot have $\sin\theta=0$, or $\cos\theta=0$; or $\theta=0$, or $\theta=\frac{\pi}{2}$; because the magnitude of the angle θ is confined within certain limits, given by the equations (178) or (179); neither can we

have $Q(\sin^2\theta - \cos^2\theta) - 2T\cos^2\theta = 0$; for if we assume the truth of this supposition, we shall find, writing $\theta_{,}$ for θ,

$$\tan^2\theta_{,} = \frac{Q-2T}{Q} \qquad \text{or } \sec^2\theta_{,} = \frac{2(Q-T)}{Q} \quad . \quad . \quad . \quad (a).$$

We must now compute the value of this expression.

Since $Q = a^2 + b^2 + c^2 - 2k^2$, and

$$T = (a^2 + b^2 + c^2 - k^2)\left[\frac{(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2 b^2 c^2}\right];$$

we get, after some reductions,

$$\frac{a^2 b^2 c^2}{k^2}(Q-T) = a^2 b^2 c^2 + (b^2 c^2 + a^2 c^2 + a^2 b^2)(a^2 + b^2 + c^2) - k^2(b^2 c^2 + a^2 c^2 + a^2 b^2)$$
$$- k^2\{a^4 + b^4 + c^4 + 2b^2 c^2 + 2a^2 c^2 + 2a^2 b^2\} + 2k^4(a^2 + b^2 + c^2) - k^6 \qquad (b).$$

Now this expression may be reduced to the symmetrical form

$$(b^2 + c^2 - k^2)(a^2 + c^2 - k^2)(a^2 + b^2 - k^2) \quad . \quad . \quad (c).$$

$$\text{whence } \sec^2\theta_{,} = \frac{2k^2(b^2+c^2-k^2)(a^2+c^2-k^2)(a^2+b^2-k^2)}{a^2 b^2 c^2(a^2+b^2+c^2-2k^2)} \quad (d).$$

The greatest value of $\sec\theta$, which the conditions of the problem admit, is given by the equation (178),

$\sec^2\Theta = \dfrac{k^2(a^2+c^2-k^2)}{a^2 c^2}$. Let the ratio of those secants be n,

we shall find that n is always greater than 1: put $\sec\theta_{,} = n\sec\Theta$;

$$\text{or, } \frac{\sec^2\theta_{,}}{\sec^2\Theta} = n^2 = \frac{2(b^2+c^2-k^2)(a^2+b^2-k^2)}{b^2(a^2+b^2+c^2-2k^2)}$$

or $n^2 = 2 - \dfrac{2(a^2-k^2)(k^2-c^2)}{b^2(a^2+b^2+c^2-2k^2)}$. As the extreme limits of k are a and c, let $k^2 = a^2 - \alpha^2$, $k^2 = c^2 + \gamma^2$; α and γ being positive quantities, which are small when compared with the axes. This expression may now be written,

$n^2 = 2 - \dfrac{2\alpha^2\gamma^2}{b^2(b^2+\alpha^2-\gamma^2)}$; or n is equal to $\sqrt{2}$ nearly, since the

second term may be neglected. We have therefore

$$\sec \theta_{,} = \sqrt{2} \sec \Theta;$$

a value of θ which cannot be admitted, since Θ is the maximum value of θ.

The only remaining factor is $\frac{d\theta}{dt}$; differentiating (186) and making $\frac{d\theta}{dt}=0$, we get, $-k^2(a^2-b^2)\sin 2\lambda=0$, an equation which is satisfied by $\lambda=0$, or $\lambda=\frac{\pi}{2}$, but these values of λ give $\theta=\Theta$, and $\theta=\Theta'$; or, *the maximum and minimum velocities of the pole of the instantaneous axis of rotation along the spiral, are at its greatest or least distances from the centre of the spiral;* as we might indeed have anticipated.

Taking the second differential of this expression,

$$-k^2(a^2-b^2)\cos 2\lambda;$$

this is negative when $\lambda=0$, and positive when $\lambda=\frac{\pi}{2}$. Therefore the velocity is a maximum when $\lambda=0$, and a minimum when $\lambda=\frac{\pi}{2}$. Or the velocity is least at the inner apside, and greatest at the outer.

SECTION VI.

CVI. We shall now proceed to determine the curves traced out by the poles of the principal axes of the body, during the motion, on an immovable concentric sphere. We shall first investigate the curve traced out by the axis c of the ellipsoid, or the C spiral, as for the sake of brevity it may be named.

Let ρ be the angle between the pole of the impressed couple, and the pole of the axis c. Then the usual formula gives us

$$\left(\frac{d\sigma}{dt}\right)^2=\left(\frac{d\rho}{dt}\right)^2+\sin^2\rho\left(\frac{d\psi}{dt}\right)^2.$$

Now, ρ being the angle between k and the axis c of the ellipsoid, $\cos\rho = \frac{z}{k}$, $\sin\rho = \frac{\sqrt{k^2 - z^2}}{k}$, $\tan\rho = \frac{\sqrt{k^2 - z^2}}{z}$.

hence $\left(\frac{d\rho}{dz}\right)^2 = \frac{1}{k^2 - z^2}$. In (154) it was shown that

$\left(\frac{dz}{dt}\right)^2 = \frac{x^2 k^2 \, \mathrm{X\,Y}}{a^2 b^2 c^4}$, and in (168) we found $\frac{d\psi}{dt} = \frac{x\,k^2}{c^2}\left(\frac{c^2 - z^2}{k^2 - z^2}\right)$.

Before we proceed further, it is proper to show that the curve has two asymptotic circles; for ι being the inclination of the vector arc to the curve, at the point of contact,

$$\tan\iota = \frac{\sin\rho \frac{d\psi}{dt}}{\frac{d\rho}{dt}} = \frac{a\,b\,(c^2 - z^2)}{\sqrt{\mathrm{X\,Y}}} \quad . \quad . \quad . \quad . \quad \text{(a)}.$$

When $\mathrm{X} = 0$, or $\mathrm{Y} = 0$, we shall have $\tan\iota = \infty$, or $\iota = \frac{\pi}{2}$.

The radii of the asymptotic circles may be found by making $\mathrm{X} = 0$, and $\mathrm{Y} = 0$, or $(a^2 - k^2)\,c^2 - (a^2 - c^2)\,z^2 = 0$,
and $(b^2 - c^2)\,z^2 - (b^2 - k^2)\,c^2 = 0$.

Resuming our equations, and making the suggested substitutions,

$\frac{a^2 b^2 c^4}{x^2 k^2}\left(\frac{d\sigma}{dt}\right)^2 = \frac{\mathrm{X\,Y} + a^2 b^2 (c^2 - z^2)^2}{(k^2 - z^2)}$; reducing this expression by the help of the preceding relations, we obtain

$$\frac{a^2 b^2 c^2}{x^2 k^2}\left(\frac{d\sigma}{dt}\right)^2 = c^2\,(a^2 + b^2 - k^2) - (a^2 + b^2 - c^2)\,z^2 \quad . \quad (215).$$

Let distances a', b', c' be assumed along the axes of the ellipsoid a, b, c, and inversely proportional to those axes; so that $a\,a' = b\,b' = c\,c' = k^2$. Let v, v', v'' be the velocities of the extremities of these lines respectively. Whence $c'\frac{d\sigma}{dt}$ will be the velocity of the extremity of c',

or $v''=c'\frac{d\sigma}{dt}=\frac{k^2}{c}\,\frac{d\sigma}{dt}$; hence $\left(\frac{d\sigma}{dt}\right)^2=\frac{c^2}{k^4}\,v''^2$. Substituting this value in the last equation, and multiplying by $a^2\,b^2$, we find

$$\frac{a^4\,b^4\,c^4}{\kappa^2\,k^6}.v''^2=a^2b^2c^2(a^2+b^2-k^2)+2\,a^2b^2c^2z^2-(a^2+b^2+c^2)a^2b^2z^2.$$

Writing analogous expressions for the other axes, and introducing the relations given by the equations of the ellipsoid and sphere, we find, on adding those expressions,

$$v^2+v'^2+v''^2=\frac{\kappa^2\,k^6}{a^2\,b^2\,c^2}\,(a^2+b^2+c^2-k^2) \quad . \quad . \quad (216).$$

We have therefore this theorem:

If right lines are taken along the three principal axes of the body from the centre, and inversely proportional to the square roots of the moments of inertia round those axes, the sum of the squares of the velocities of their extremities is constant during the motion.

CVII. Let us now resume the general equation, and proceed to find the lengths of the spirals, traced by the principal axes during the motion. The equation for c is, as in (215),

$$\frac{a^2\,b^2\,c^2}{\kappa^2\,k^2}\left(\frac{d\sigma}{dt}\right)^2=c^2\,(a^2+b^2-k^2)-(a^2+b^2-c^2)\,z^2 \quad . \quad (a).$$

Assume as in (155),

$$z^2=\frac{(a^2-k^2)\,(b^2-k^2)\,c^2}{(a^2-k^2)(b^2-c^2)\cos^2\phi+(b^2-k^2)\,(a^2-c^2)\sin^2\phi},$$

and substitute this value of z in (a); we shall then have

$$\frac{a^2\,b^2}{\kappa^2\,k^2(k^2-c^2)}\left(\frac{d\sigma}{dt}\right)^2=\frac{[a^2\,(a^2-k^2)\cos^2\phi+b^2(b^2-k^2)\sin^2\phi]}{[(a^2-k^2)\,(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi]} \quad . \quad . \quad (b)$$

and $$\left(\frac{dt}{d\phi}\right)^2=\frac{a^2\,b^2\,c^2}{\kappa^2\,k^2[(a^2-k^2)\,(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi]}\text{ as in (155) (c)}$$

Whence

$$\frac{1}{c\sqrt{k^2-c^2}}\left(\frac{d\sigma}{d\phi}\right)=\frac{\sqrt{a^2(a^2-k^2)\cos^2\phi+b^2(b^2-k^2)\sin^2\phi}}{(a^2-k^2)\,(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi} \quad . \quad . \quad . \quad (d).$$

$$\left.\begin{array}{ll}\text{Let A}=a^2\,(a^2-k^2), & \text{C}=(a^2-k^2)\,(b^2-c^2)\\ \text{B}=b^2\,(b^2-k^2), & \text{D}=(b^2-k^2)\,(a^2-c^2)\end{array}\right\} \quad . \quad (e).$$

Then $\dfrac{1}{c\sqrt{k^2-c^2}}\left(\dfrac{d\sigma}{d\phi}\right) = \dfrac{\text{A}\cos^2\phi + \text{B}\sin^2\phi}{\text{C}\cos^2\phi + \text{D}\sin^2\phi} \times \dfrac{1}{\sqrt{\text{A}\cos^2\phi + \text{B}\sin^2\phi}}$ . . (f)

and this expression may be transformed into

$$\frac{1}{c\sqrt{k^2-c^2}}\left(\frac{d\sigma}{d\phi}\right) = \frac{\text{BC}-\text{AD}}{\text{C}(\text{C}-\text{D})\sqrt{\text{A}}}\,\frac{1}{\left[1-\left(\frac{\text{C}-\text{D}}{\text{C}}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{\text{A}-\text{B}}{\text{A}}\right)\sin^2\phi}}$$

$$+\frac{\text{A}-\text{B}}{(\text{C}-\text{D})\sqrt{\text{A}}}\,\frac{1}{\sqrt{1-\left(\frac{\text{A}-\text{B}}{\text{A}}\right)\sin^2\phi}} \quad . \quad . \quad . \quad . \quad (217).$$

Equations (e) give us

$$\left.\begin{array}{ll} \dfrac{\text{BC}-\text{AD}}{\text{C}(\text{C}-\text{D})} = \dfrac{-(a^2+b^2-c^2)(b^2-k^2)}{(b^2-c^2)(k^2-c^2)}, & \dfrac{\text{A}-\text{B}}{\text{C}-\text{D}} = \dfrac{a^2+b^2-k^2}{k^2-c^2} \\ \dfrac{\text{A}-\text{B}}{\text{A}} = \dfrac{(a^2-b^2)(a^2+b^2-k^2)}{a^2(a^2-k^2)}, & \dfrac{\text{C}-\text{D}}{\text{C}} = \dfrac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)} \end{array}\right\} \cdot \text{(g)}.$$

Now $e'^2 = \dfrac{\tan^2\alpha' - \tan^2\beta'}{\tan^2\alpha'}$; as in (12). Substituting the values of $\tan\alpha'$, $\tan\beta'$ given in (207), we get

$e'^2 = \dfrac{(a^2-b^2)(a^2+b^2-k^2)}{a^2(a^2-k^2)} = \dfrac{\text{A}-\text{B}}{\text{A}}$; hence e' is the modulus.

In (156) it was shown that $\sin^2\varepsilon = \dfrac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)} = \dfrac{\text{C}-\text{D}}{\text{C}}$; whence $\sin^2\varepsilon$ is the parameter. Making these substitutions, we obtain the resulting equation,

$$\sigma = \sqrt{\frac{c^2(k^2-c^2)}{a^2(a^2-k^2)}}\left(\frac{a^2+b^2-k^2}{k^2-c^2}\right)\int\frac{d\phi}{\sqrt{1-e'^2\sin^2\phi}}$$

$$-\sqrt{\frac{c^2(k^2-c^2)}{a^2(a^2-k^2)}}\left(\frac{(a^2+b^2-c^2)(b^2-k^2)}{(k^2-c^2)(b^2-c^2)}\right)\int\frac{d\phi}{[1-\sin^2\varepsilon\sin^2\phi]\sqrt{1-e'^2\sin^2\phi}} \quad (218).$$

As $\sin^2\varepsilon$ is less than e'^2, this elliptic function is of the third order and *logarithmic* form. That it is so, may be shown by constructing the expression $(1+n)\left(1+\dfrac{c^2}{n}\right)$; or in this case

$\cos^2 \varepsilon \left(1 - \frac{e'^2}{\sin^2 \varepsilon}\right) = \cot^2 \varepsilon (\sin^2 \varepsilon - e'^2)$; whence the *criterion*

of *circularity* becomes $- \frac{(b^2-k^2)^2 (a^2-c^2)(a^2+b^2-c^2)}{a^2(a^2-k^2)(k^2-c^2)(b^2-c^2)}$ (k).

This is a quantity essentially negative, whatever be the value we assign to k between its limits a and c. Hence the polar spirals described during the motion by the greatest or the least principal axes, are rectified by elliptic functions of the third order and *logarithmic* form.

When the ellipsoid is one of revolution, the elliptic function may be reduced from the third order to a circular arc. In this case $a=b$, whence $\sin \varepsilon = 0$, $e'=0$.

Adding together the coefficients of the integrals, now become identical, we get

$$\sigma = \frac{a^2}{a^2 - c^2} \tan \alpha' \phi \quad . \quad . \quad . \quad . \quad . \quad (219).$$

CVIII. Multiply equation (218) by the expression

$\frac{(a^2 - b^2)\sqrt{a^2+b^2-c^2}}{a\,b\,c}$, which depends solely on the moments of

inertia of the body. Let m be written for this factor; then (218) will become

$$\left.\begin{aligned} m\sigma = {} & \frac{(a^2-b^2)(a^2+b^2-k^2)\sqrt{a^2+b^2-c^2}}{a^2 b\sqrt{(a^2-k^2)(k^2-c^2)}} \int \frac{d\phi}{\sqrt{1-e'^2 \sin^2\phi}} \\ & - \frac{(a^2-b^2)(b^2-k^2)(a^2+b^2-c^2)^{\frac{3}{2}}}{a^2 b(b^2-c^2)\sqrt{(a^2-k^2)(k^2-c^2)}} \int \frac{d\phi}{[1-\sin^2\varepsilon \sin^2\phi]\sqrt{1-e'^2\sin^2\phi}} \end{aligned}\right\} \quad (220).$$

Now ε is the focal angle of the invariable cone, and e' is the eccentricity of the plane base of the cone of rotation. Let there be a cone which shall have the same focal lines as the invariable cone, and a plane elliptic base similar to that of the cone of rotation. Then $\alpha_{,}$ and $\beta_{,}$ being the principal angles of such a cone, we shall have, see (12) and (13),

$$\frac{\tan^2 \alpha_{,} - \tan^2 \beta_{,}}{\tan^2 \alpha_{,}} = e'^2, \text{ and } \frac{\cos^2 \beta_{,} - \cos^2 \alpha_{,}}{\cos^2 \beta_{,}} = \sin^2 \varepsilon \quad . \quad . \quad (a).$$

$$\text{or } \tan^2\alpha_{,}=\frac{a^2(k^2-c^2)}{(b^2-k^2)(a^2+b^2-c^2)},\ \tan^2\beta_{,}=\frac{b^2(k^2-c^2)}{(a^2-k^2)(a^2+b^2-c^2)}\ \text{(b).}$$

$$\text{Whence } \cos^2\alpha_{,}=\frac{(a^2+b^2-c^2)(b^2-k^2)}{(a^2+b^2-k^2)(b^2-c^2)}\quad .\quad .\quad .\ \text{(c).}$$

$$e_{,}^2=e'^2=\frac{(a^2-b^2)(a^2+b^2-k^2)}{a^2(a^2-k^2)}.\ \sin^2\epsilon_{,}=\sin^2\epsilon=\frac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}\ \text{(d).}$$

By the help of those relations, if we construct the expression $\frac{e_{,}^2}{\tan\beta_{,}}$ we shall find it to be equal to the coefficient of the elliptic function of the *first* order in the equation (220). In like manner if we construct the expression $\frac{e_{,}^2}{\tan\beta_{,}}\cos^2\alpha$, we shall obtain the coefficient of the elliptic function of the *third* order in the same equation. Accordingly (220) may be written,

$$m\sigma=\frac{e_{,}^2}{\tan\beta_{,}}\int\frac{d\phi}{\sqrt{1-e_{,}^2\sin^2\phi}}-\frac{e_{,}^2}{\tan\beta_{,}}\cos^2\alpha_{,}\int\frac{d\phi}{[1-\sin^2\epsilon_{,}\sin^2\phi]\sqrt{1-e_{,}^2\sin^2\phi}}\ (221).$$

Now if we compare this formula with (75), we shall find them identical. Whence we infer, that the length of the spiral described by the pole of the greatest or the least axes of the ellipsoid on a fixed sphere,—*the semidiameter k being the next in the order of magnitude to such greatest or least axe*—will be equal to the length of the curve there defined, as generated on the surface of a sphere, according to a given law.

In that definition, we assumed $\cos i = m\sin\psi\cos\psi$. i' being the corresponding angle in the spherical ellipse, it is not difficult to show that $\cos i' = \sin\eta\tan\eta\tan\rho\sin\psi\cos\psi$. η being the angle between the axis and the circular sections of the cone.

CIX. On the spiral described by the pole of the greater principal axis, or the A spiral.

In the general equation (215) substitute x for z, and interchange a and c; we shall then have

$$\frac{a^2b^2c^2}{x^2k^2}\left(\frac{d\sigma'}{dt}\right)^2=a^2(c^2+b^2-k^2)-(c^2+b^2-a^2)x^2\quad .\quad .\quad \text{(a).}$$

In (159) we found

$$x^2 = \frac{a^2(b^2-k^2)(k^2-c^2)\sin^2\phi}{(a^2-k^2)(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi}.$$

Substituting this value of x^2 in the preceding equation, and introducing the value of $\left(\frac{dt}{d\phi}\right)$ given in (155), we obtain the resulting equation,

$$\frac{\frac{d\sigma'}{dt}}{a\sqrt{a^2-k^2}} = \frac{\sqrt{(b^2-c^2)(c^2+b^2-k^2)\cos^2\phi+b^2(b^2-k^2)\sin^2\phi}}{(a^2-k^2)(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi} \quad \text{(b.)}$$

This expression may be reduced in the same way as (d), in (CVII), omitting the steps for the sake of brevity. The resulting equation will be found

$$\left.\begin{aligned}\sigma' = {} & \frac{c^2}{a^2-b^2}\left[\frac{a^2(a^2-k^2)}{(b^2-c^2)(b^2+c^2-k^2)}\right]^{\frac{1}{2}}\int\frac{d\phi}{\sqrt{1-\sin^2\alpha'\sin^2\phi}} \\ & -\frac{(b^2-k^2)(b^2+c^2-a^2)}{(a^2-b^2)(a^2-k^2)}\left[\frac{a^2(a^2-k^2)}{(b^2-c^2)(b^2+c^2-k^2)}\right]^{\frac{1}{2}}\int\frac{d\phi}{[1-\sin^2\epsilon\sin^2\phi]\sqrt{1-\sin^2\alpha'\sin^2\phi}}\end{aligned}\right\} \quad (222).$$

An elliptic integral which is also of the third order and *logarithmic* form.

The parameter is the square of the sine of the semifocal angle of the invariable cone, while the modulus is the sine of the major principal arc of the cone of rotation.

When $a=b$, $\sin\epsilon=0$, and the above expression assumes the form,

$$\sigma' = \left(\frac{a^2+c^2-k^2}{a^2-k^2}\right)\cos\alpha'\int\frac{d\phi}{\sqrt{1-\sin^2\alpha'\sin^2\phi}} \quad . \quad (223).$$

In (XXIII.) it was shown that $\cos\alpha'\int\frac{d\phi}{\sqrt{1-\sin^2\alpha'\sin^2\phi}}$ is the algebraical representative of an arc of the spherical parabola whose major principal arc $\alpha_{,}$ is given by the equation

$$\tan^2\alpha_{,} = \frac{1+\cos\alpha'}{1-\cos\alpha'} = \frac{1}{\tan^2\frac{\alpha'}{2}}; \text{ whence } \alpha_{,}+\frac{\alpha'}{2} = \frac{\pi}{2},$$

or α' and $2\,\alpha_{,}$ are supplemental.

CX. On the spiral described by the mean axis b of the ellipsoid, or the *mean* or B spiral.

In the general equation (215), interchanging b and c, also y and z, we obtain the result

$$\frac{a^2 b^2 c^2}{x^2 k^2}\left(\frac{d\,\sigma''}{d\,t}\right)^2 = b^2(a^2+c^2-k^2)-(a^2+c^2-b^2)\,y^2 \quad . \quad . \quad \text{(a)}.$$

for y^2 substitute its value given in (159),

$$y^2 = \frac{b^2\,(a^2-k^2)\,(k^2-c^2)\cos^2\varphi}{(a^2-k^2)\,(b^2-c^2)\cos^2\varphi+(b^2-k^2)\,(a^2-c^2)\sin^2\varphi}.$$

Introducing the value of $\left(\dfrac{d\,t}{d\,\varphi}\right)$ found in (155), we shall obtain

$$\frac{1}{b\sqrt{b^2-k^2}}\left(\frac{d\sigma''}{d\,\varphi}\right)=\frac{\sqrt{a^2(a^2-k^2)\cos^2\varphi+(a^2-c^2)(a^2+c^2-k^2)\sin\,\varphi}}{(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi}\quad\text{(b)}.$$

$$\left.\begin{aligned}&\text{Let } \mathrm{A}=a^2\,(a^2-k^2), && \mathrm{C}=(a^2-k^2)\,(b^2-c^2),\\ &\mathrm{B}=(a^2-c^2)\,(a^2+c^2-k^2), && \mathrm{D}=(b^2-k^2)\,(a^2-c^2).\end{aligned}\right\}\ \text{(c)}.$$

And the preceding equation may be written

$$\frac{1}{b\sqrt{b^2-k^2}}\left(\frac{d\sigma''}{d\,\varphi}\right)=\frac{\mathrm{A}\cos^2\varphi+\mathrm{B}\sin^2\varphi}{\mathrm{C}\cos^2\varphi+\mathrm{D}\sin^2\varphi}\quad\frac{1}{\sqrt{\mathrm{A}\cos^2\varphi+\mathrm{B}\sin^2\varphi}}\quad\text{(d)};$$

or as B > A, this equation may be transformed into

$$\left.\begin{aligned}\frac{\sqrt{\mathrm{B}}}{b\sqrt{b^2-k^2}}\left(\frac{d\,\sigma''}{d\phi}\right)=\frac{\mathrm{BC}-\mathrm{AD}}{\mathrm{D(C-D)}}\int\frac{d\phi}{\left[1+\left(\frac{\mathrm{C-D}}{\mathrm{D}}\right)\cos^2\phi\right]\sqrt{1-\left(\frac{\mathrm{B-A}}{\mathrm{B}}\right)\cos^2\phi}}\\ -\frac{(\mathrm{B-A})}{\mathrm{C-D}}\int\frac{d\phi}{\sqrt{1-\left(\frac{\mathrm{B-A}}{\mathrm{B}}\right)\cos^2\phi}}\end{aligned}\right\}\ (224).$$

If we now compute the value of the coefficients in this equation by the help of (c), we shall find, $2\,\epsilon$ being the focal angle of the invariable cone, as shown in (156),

$$\frac{\mathrm{C-D}}{\mathrm{D}}=\frac{(a^2-b^2)\,(k^2-c^2)}{(a^2-c^2)\,(b^2-k^2)}=\tan^2\epsilon \quad . \quad . \quad . \quad \text{(e)}.$$

$\frac{\text{B}-\text{A}}{\text{B}}=\frac{c^2(k^2-c^2)}{(a^2-c^2)(a^2+c^2-k^2)}=\sin^2\beta'$. β' being the lesser principal angle of the cone of rotation, as in (207). We have also

$\frac{\text{BC}-\text{AD}}{\text{D}(\text{C}-\text{D})}=\frac{(a^2-k^2)}{(b^2-k^2)}\frac{(a^2+c^2-b^2)}{(a^2-b^2)}$, and $\sqrt{\text{B}}=\sqrt{(a^2-c^2)(a^2+c^2-k^2)}$

making those substitutions, (224) becomes

$$\sigma''=\frac{b(a^2-k^2)(a^2+c^2-b^2)}{(a^2-b^2)\sqrt{(a^2-c^2)(b^2-k^2)(a^2+c^2-k^2)}}\int\frac{d\phi}{[1+\tan^2\epsilon\cos^2\phi]\sqrt{1-\sin^2\beta'\cos^2\phi}}$$

$$-\frac{c^2b\sqrt{b^2-k^2}}{(a^2-b^2)\sqrt{(a^2-c^2)(a^2+c^2-k^2)}}\int\frac{d\phi}{\sqrt{1-\sin^2\beta'\cos^2\phi}} \quad . \quad . \quad . \quad (225).$$

As the parameter $\tan^2\epsilon$ is positive, the elliptic integral of the third order is of the *circular* form.

When $a=b$, $\tan\epsilon=0$, and the elliptic integral of the third order in the preceding equation is reduced to the first. Adding the above expressions together, and reducing,

$$\sigma''=\left(\frac{a^2+c^2-k^2}{a^2-k^2}\right)\cos\alpha'\int\frac{d\phi}{\sqrt{1-\sin^2\alpha'\cos^2\phi}} \quad . \quad . \quad (a).$$

This expression agrees with the one found for the greater spiral, differing from it only in the amplitude, which is complementary.

We shall now proceed to eliminate from (225) the integral of the first order.

Multiply this equation by the factor $\sqrt{\frac{(a^2-b^2)(b^2-c^2)}{b^2(a^2+c^2-b^2)}}$.

Let as before $\alpha_{,}$ and $\beta_{,}$ be the principal semi-angles of a cone whose focals shall coincide with those of the invariable cone, and the planes of whose circular sections shall make the angles β' with the internal axe; then assuming the equations established in (13) and (e), we shall have

$\frac{\tan^2\alpha_{,}-\tan^2\beta_{,}}{\sec^2\beta_{,}}=\tan^2\epsilon_{,}=\tan^2\epsilon=\frac{(a^2-b^2)(k^2-c^2)}{(a^2-c^2)(b^2-k^2)}$, and

$$\frac{\sin^2\alpha_{,}-\sin^2\beta_{,}}{\sin^2\alpha_{,}}=\sin^2\eta_{,}=\sin^2\beta'=\frac{c^2(k^2-c^2)}{(a^2-c^2)(a^2+c^2-k^2)}$$

as in (207); whence, making the substitutions indicated,

$$\tan^2\alpha_{,}=\frac{(a^2-b^2)(a^2+c^2-k^2)}{c^2(b^2-k^2)},\qquad \tan^2\beta_{,}=\frac{a^2(a^2-b^2)}{c^2(b^2-c^2)}\quad .\ \text{(b)}.$$

by the help of those equations, we may show that

$$\frac{\cos\beta_{,}}{\cos\alpha_{,}\sin\alpha_{,}}=\frac{(a^2-k^2)\sqrt{(b^2-c^2(a^2+c^2-b^2)}}{\sqrt{(a^2-c^2)(a^2-b^2)(b^2-k^2)(a^2+c^2-k^2)}}\quad \text{(c)}.$$

$$\text{and}\quad \frac{\cos\beta_{,}\cos\alpha_{,}}{\sin\alpha_{,}}=\frac{c^2\sqrt{(b^2-k^2)(b^2-c^2)}}{\sqrt{(a^2-c^2)(a^2-b^2)(a^2+c^2-k^2)(a^2+c^2-b^2)}}\text{(d)}.$$

Whence (225) may now be written

$$\left.\begin{aligned}\left(\frac{(a^2-b^2)(b^2-c^2)}{b^2(a^2+c^2-b^2)}\right)^{\frac{1}{2}}\sigma''=\frac{\cos\beta_{,}}{\cos\alpha_{,}\sin\alpha_{,}}\int\frac{d\phi}{[1+\tan^2\varepsilon\cos^2\phi]\sqrt{1-\sin^2\eta_{,}\cos^2\phi}}\\ -\frac{\cos\beta_{,}\cos\alpha_{,}}{\sin\alpha_{,}}\int\frac{d\phi}{\sqrt{1-\sin\ \eta_{,}\cos^2\phi}}\end{aligned}\right]\ (226).$$

If now we turn to (44), in which elliptic integrals are compared, having the same amplitude, but positive and negative parameters respectively, we shall find it identical with the preceding equation; which may now therefore be written

$$\left[\frac{(a^2-b^2)(b^2-c^2)}{b^2(a^2+c^2-b^2)}\right]^{\frac{1}{2}}\sigma''=\frac{\tan\beta_{,}}{\tan\alpha_{,}}\sin\beta_{,}\int\frac{d\phi}{[1-e_{,}^2\sin\ \phi]\sqrt{1-\sin^2\eta_{,}\cos^2\phi}}$$

$$+\tan^{-1}\left[\frac{e_{,}\tan\varepsilon_{,}\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta_{,}\cos^2\phi}}\right]\quad .\quad .\quad .\quad (227).$$

If we take the complete function, the circular arc vanishes. We may therefore conclude that the length of the mean spiral, or of the spiral described by the pole of the mean axe b of the ellipsoid, between any two of its asymptotic positions, is equal to a quadrant of a spherical ellipse. The cone of which this spherical ellipse is the base, may with ease be determined. It must have the same focal lines as the invariable cone; and its minor principal arc is the angle between the cyclic diameters of the ellipsoid; for the cyclic diameter whose square is $a^2+c^2-b^2$

makes with the axis c an angle the square of whose tangent is

$$\frac{a^2(a^2-b^2)}{c^2(b^2-c^2)}. \text{ In (f) we found } \tan^2\beta_{,}=\frac{a^2(a^2-b^2)}{c^2(b^2-c^2)},$$

or $2\beta_{,}$ is the angle between the cyclic diameters of the ellipsoid.

We have thus investigated the equations of the spirals described on a fixed concentric sphere by the three principal axes of a body, which we have named the *greater*, *mean*, and *lesser*, or the A, B, and C spirals. It is not a little remarkable that the rectification of the greater and lesser spirals must be effected by elliptic integrals of the third order and *logarithmic* form, while the rectification of the mean spiral depends on an elliptic integral of the third order and *circular* form. It will moreover be evident, on referring to (210), that the elliptic integrals which express the lengths of the spirals described by the instantaneous axis of rotation and the mean principal axe of the body have the same amplitude, and are each of the *circular* form; while the integrals which determine the spirals described by the greatest and the least principal axes of the body, also have the *same* amplitude, which is complementary to the former, and are of the *logarithmic* form.

CXI. We may determine the maximum and minimum velocities with which the poles of the principal axes of the body describe their respective spirals on the fixed concentric sphere. Resuming the equation of the spirals traced by the principal axes,

$$\frac{a^2b^2c^2}{x^2k^2}\left(\frac{d\sigma}{dt}\right)^2=c^2(a^2+b^2-k^2)-(a^2+b^2-c^2)z^2; \text{ differentiating}$$

and putting the differential equal to cypher, we get $\frac{dz}{dt}=0$.

It was shown in (121) that $\frac{dz}{dt}=\frac{f(a^2-b^2)xy}{a^2b^2}$.

This is $=0$, whenever the position of the axis k renders $x=0$, or $y=0$; and as k is at its greatest or least distance from the axis c of the ellipsoid, whenever it lies in one of the principal planes, the velocity of the pole of c on the spiral is the greatest or the least, whenever the axis c is at its greatest or least distance from the axis k.

The same proof may be applied to determine the extreme velocities of the poles of a and b.

CXII. When the axis k coincides absolutely with one of the principal axes, the equations which give the lengths of the spirals change their form. Let k coincide with c, the least or the greatest of the principal axes of the body; the spirals described by the principal axes are equivalent to circular arcs. If $k=c$, (218), (222), and (224) become

$$\sigma = 0,\ \sigma' = \frac{ab\phi}{\sqrt{(a^2-c^2)(b^2-c^2)}},\ \sigma'' = \frac{ab\phi}{\sqrt{(a^2-c^2)(b^2-c^2)}}.$$

SECTION VII.

CXIII. There are two particular cases of the general problem which require separate investigations. When the plane of the impressed couple is at right angles to, or coincides with, the plane of one of the circular sections of the ellipsoid of moments.

We shall first take the case when the plane of the impressed couple is at right angles to the plane of one of the circular sections of the ellipsoid, or $k=b$. If we introduce this value of k into the equation of the invariable cone in (LXVII) we shall obtain the following equation:

$$c^2(a^2-b^2)x^2+a^2(c^2-b^2)z^2=0 \quad . \quad . \quad (228).$$

This expression is the equation of the two plane circular sections of the ellipsoid, which intersect in the mean axe. If then, to fix our ideas, we conceive the plane of the impressed couple to be horizontal, one of the circular sections of the ellipsoid will be vertical during the motion.

To determine in this case the locus of the instantaneous axis of rotation in the body. If we write b for k in the equation of the cone of rotation (134), we get,

$$a^2(a^2-b^2)x^2+c^2(c^2-b^2)z^2=0 \quad . \quad . \quad (229).$$

The equation of two plane sections of the ellipsoid passing through the mean axe, and perpendicular to the umbilical diameters of the ellipsoid.

We perceive therefore that the axis of the impressed couple, and the instantaneous axis of rotation, describe planes in the body during the motion.

To find the greatest elongation of the axis of rotation from the axis k. This is nothing more than to find the angle which a perpendicular from the centre, on a tangent passing through the vertex of k or b, makes with it, in an ellipse where semiaxes are a and c. Now h being the conjugate diameter to k or b, and P the perpendicular on the tangent,

$$h^2+b^2=a^2+c^2, \text{ and } \mathrm{P}h=ac. \quad \text{Let this angle be } \vartheta.$$

$$\text{Then } \tan^2\vartheta=\frac{b^2-\mathrm{P}^2}{\mathrm{P}^2}=\frac{(a^2-b^2)\,(b^2-c^2)}{a^2c^2}=l^2 \quad . \quad . \quad (230).$$

CXIV. To determine the time.

In the general equation (153) let $k=b$, and we shall find,

$$\frac{dt}{dz}=\frac{abc^2}{fz\sqrt{b^2-c^2}\sqrt{c^2(a^2-b^2)-(a^2-c^2)z^2}} \quad . \quad . \quad (a).$$

Assume $\qquad (a^2-c^2)\,z^2=c^2\,(a^2-b^2)\sin^2\varphi. \quad . \quad (231).$

In which φ is the angle between k and the mean axe of the ellipsoid, measured on a circular section of the surface. By this transformation, equation (a) may be changed into

$$\frac{dt}{d\varphi}=\frac{ac}{\varkappa\sqrt{(a^2-b^2)\ (b^2-c^2)}}\left(\frac{1}{\sin\varphi}\right) \quad . \quad . \quad (232).$$

It was shown in (230) that $\tan\vartheta$, the maximum value of $\tan\theta=\sqrt{\dfrac{(a^2-b^2)\,(b^2-c^2)}{a^2c^2}}=l$. Let $i=\varkappa l$; the preceding equation, when integrated, will become, putting C for the constant,

$$it=\log.\tan\frac{\varphi}{2}+\mathrm{C} \quad . \quad . \quad . \quad (233).$$

To determine the value of this constant. Let δ be the initial distance of the pole of k from the axis b, at the beginning of the motion: then $0=\log.\tan\dfrac{\delta}{2}+\mathrm{C}$. Subtracting we shall have

$$it = \log. \left\{ \frac{\tan \frac{\varphi}{2}}{\tan \frac{\delta}{2}} \right\} \quad . \quad . \quad . \quad (c).$$

Let $\tan \frac{\delta}{2} = m$, and the last equation may be written

$$\tan \frac{\varphi}{2} = m e^{it}, \text{ or as } i = \varkappa l, \ \tan \frac{\varphi}{2} = m e^{\varkappa l t}. \ . \quad (234).$$

e being the base of the Neperian logarithms.

When δ is absolutely equal to 0, m also is equal to 0, and φ is 0, however large the value we may assign to the time t. But when δ is only very small, m will be a very small quantity, and therefore t must be very large before its magnitude can have any appreciable effect on the magnitude of φ. Hence the pole of k will diverge slowly from the mean axe b. When the initial distance δ is supposed to be considerable, then m is no longer an indefinitely small quantity, and a small increase in t will produce a considerable effect in the magnitude of φ.

Again: let the axis of the impressed couple, by the motion of the semicircular section passing through it, be approximated to indefinitely, by the prolongation of the principal axis b, within a very small angle δ'.

Let T be the future time at which the prolongation of the axis b shall arrive within a certain small angle δ' of k. Then $\varphi = \pi - \delta'$; and $iT = \log. \tan \left(\frac{\pi}{2} - \frac{\delta'}{2} \right) + C$. As the initial distance of b from k must be supposed as before to be δ,

$0 = \log. \tan \left(\frac{\delta}{2} \right) + C$. Whence $-iT = \log. \left\{ \tan \left(\frac{\delta}{2} \right) \tan \left(\frac{\delta'}{2} \right) \right\}$.

Let $m = \tan \left(\frac{\delta}{2} \right)$ as before, then

$$m \tan \frac{\delta'}{2} = e^{-iT} \quad . \quad . \quad . \quad . \quad (235).$$

In this equation T will be infinite on two suppositions, either $m = 0$, or $\tan \frac{\delta'}{2} = 0$. The former shows that T will be infinite

if b never departs *from* coincidence with the axis of the impressed couple. From the second we may infer that b never can, having once departed *from* coincidence with k, again coincide *with* it.

We may therefore infer that the motion of k in the body will be as follows. When the coincidence of k with the mean axe is disturbed, and the disturbance takes place along one or other of the circular sections of the ellipsoid, the axis b at first diverges very slowly from k; then with greater rapidity, until this velocity reaches a maximum state. The velocity then decreases, so that b, with a motion continually retarded, approaches indefinitely near to, without ever absolutely reaching, the axis of the impressed couple.

CXV. To find the value of θ the angle between the axis of rotation and the axis of the plane of the impressed couple.

In (120) writing b for k, and $c^2(a^2-b^2)\sin^2\phi$ for $(a^2-c^2)z^2$, we obtain $\tan\theta = l\sin\phi$. Hence θ varies from its inferior limit to ϑ, as ϕ varies from δ to $\frac{\pi}{2}$.

It was shown in (116) that the velocity of the pole of the plane of the impressed couple along the invariable conic was $f\tan\theta$. Writing v for this velocity, $\text{v} = b\varkappa l\sin\phi$. (236).

As $\tan\theta = l\sin\phi$, $\omega = \frac{f}{\text{P}}$, $\tan^2\theta = \frac{k^2-\text{P}^2}{\text{P}^2}$, ω being the angular velocity, whence $\omega^2 = \varkappa^2\{1 + l^2\sin^2\phi\}$, or $\omega = \varkappa\sec\theta$. (237).

CXVI. To determine the angle ψ which the line of the nodes makes with a fixed line in the plane of the impressed couple.

Resuming the equation (168), putting b for k, and

$c^2(a^2-b^2)\sin^2\phi$ for $z^2(a^2-c^2)$, as in (231), we get,

$$\frac{z^2}{b^2-z^2} = \frac{\tan^2\eta\sin^2\phi}{1+\tan^2\eta\sin^2\phi}.$$ Writing $\tan^2\eta$ for $\frac{c^2(a^2-b^2)}{a^2(b^2-c^2)}$,

which represents the tangent of half the dihedral angle between the circular sections of the ellipsoid, or half the angle between the cyclic axes. We also have

$\frac{dt}{d\phi}=\frac{1}{\varkappa l \sin\phi}$, as in (232). Making these substitutions in the

equation (168), $\psi = -\varkappa t + \varkappa\left(\frac{k^2-c^2}{c^2}\right)\int\frac{z^2\,dt}{k^2-z^2}$; we find

$$\psi = -\varkappa\, t + \tan^{-1}[\tan\eta\cos\phi] + \text{constant.} \quad . \quad . \quad (238).$$

To determine this constant.

At the beginning of the motion let the axis of the plane of the impressed couple very nearly coincide with the mean axis of the ellipsoid. Then ϕ is very small, and $\cos\phi$ very nearly equal to 1: we thus get $0=\tan^{-1}(\tan\eta)+C$, or $C=-\eta$.

hence $\psi = -\varkappa\, t + \tan^{-1}(\tan\eta\cos\phi) - \eta \quad . \quad . \quad . \quad (239).$

The limits of ϕ are 0 and π, between which limits the pole of the impressed couple lies during the motion. Now when $\phi=0$, $\cos\phi=1$, and $\tan^{-1}(\tan\eta\cos\phi)=\eta$. When $\phi=\pi$, $\cos\phi=-1$, and $\tan^{-1}\{\tan\eta\times-1\}=-\eta$. Whence $-\psi=\varkappa\,\mathrm{T}+2\eta$ (240).

Writing T for the period in which the semicircle is described by k.

Thus we perceive that the infinite angle ψ is made up of two parts, one of which increases as the time, while the other continually approximates to a fixed limit 2η. 2η being the angle between the cyclic axes of the surface.

The geometrical interpretation of this formula it is not difficult to discover. In (LXXXVII.) it was shown that the angle ψ was made up of two parts, one of which $\varkappa t$ increases as the time, while the other may be represented by an arc of the spherical ellipse, generated by the cone supplemental to the invariable cone. As the circular sections of this latter coincide in direction with the circular sections of the ellipsoid, the cyclic axes of this latter surface will coincide with the focals of the supplemental cone. Hence, as before mentioned, the whole motion of the body may be represented by conceiving this supplemental cone to roll

without sliding on the plane of the impressed couple, while this plane revolves uniformly round its axis. But when the plane, as in this case, passes through one of the cyclic axes of the ellipsoid, this supplemental cone degenerates into a plane sector of a circle, the angle between whose bounding diameters is $2\,\eta$. Now when the plane of the impressed couple is slightly disturbed from coincidence with the plane of this circular sector, (for when k coincides with b, the plane of the impressed couple coincides with the principal plane $a\,c$, which contains the cyclic axes), it will revolve round a right line — one of the cyclic axes bounding the circular sector,—instead of rolling upon a conical surface; and this right line —the cyclic axe of the ellipsoid, or the focal of the rolling cone—becomes, in the ultimate state of this cone, the *edge* of the circular sector. The plane of $a\,c$, being disturbed from a state of coincidence with the plane of the impressed couple, will revolve round one of the cyclic axes until it approximates indefinitely on *its other side* to this plane.

Now if, instead of the cone, we imagine the sector of the circle to revolve upon the plane, the line of contact with the plane will no longer *advance continuously* upon this plane, but per saltum, starting forward through an angle $2\,\eta$ at each half revolution. So that if we imagine a number of semi-revolutions to occur, the line of contact of this sector with the plane would advance through the angles $2\,\eta$, $4\,\eta$, &c. From the nature of this motion, however, we can have but half a revolution, and even that only as a limit. It follows, therefore, that when half the semicircle is completed, or when the axis of the plane of the impressed couple comes into the plane of $a\,c$, that an angle η must at once be added to the angle ψ, or that the line of the nodes starts forward through the angle η.

CXVII. We shall now investigate the nature of the spiral, described by the pole of the instantaneous axis of rotation, in the case when $k = b$.

The spherical polar coordinates of this spiral are θ and χ.

They are connected as follows:

In general $\chi = \varkappa\, k^2 \int \frac{dt}{u^2}$, as shown in (182): put b for k in

the equation (192*) which determines u, and we shall have $u=b$, hence

$$\chi=\varkappa\, t. \quad . \quad . \quad . \quad . \quad (241).$$

This equation shows that the motion of the radius vector arc θ is uniform, being proportional to the time.

It was shown in (234) that $\tan\frac{\varphi}{2}=m\mathrm{e}^{\varkappa lt}$: writing χ for $\varkappa\, t$,

we get $\tan\frac{\varphi}{2}=m\,\mathrm{e}^{l\chi}$, and $\tan\theta=l\sin\varphi \; . \; . \; . \; . \; . \; . \; . \; (242).$

These are the equations of the spiral. We must eliminate φ from those equations.

As $\sin\varphi=2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}=2\tan\frac{\varphi}{2}\cos^2\frac{\varphi}{2}=\dfrac{2\tan\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}$,

we get $\tan\theta=\dfrac{2\,m\,l\,\mathrm{e}^{l\chi}}{1+m^2\mathrm{e}^{2l\chi}}$, or $\tan\theta=\dfrac{2\,l}{(m\,\mathrm{e}^{l\chi})+(m\,\mathrm{e}^{l\chi})^{-1}}$ (243).

A relation between the variables θ and χ, consequently the equation of the spiral.

CXVIII. The rumb line may be defined as the curve on the surface of a sphere which cuts all the meridians in a given angle. Let this constant angle be the complement of ϑ, then its cotangent is l, φ and χ being the polar spherical coordinates of

the curve; therefore $\quad l\sin\varphi=\dfrac{d\,\varphi}{d\,\chi} \quad . \quad . \quad . \quad (244).$

This is the equation of the rumb line.

Taking the integral of this equation, $\log.\tan\frac{\varphi}{2}=l\chi+\mathrm{C}$.

Let the value of φ be δ, when $\chi=0$. Then $\log.\tan\frac{\delta}{2}=\mathrm{C}$,

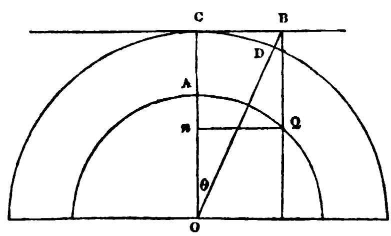
and $\tan\frac{\delta}{2}=m$, hence

$$l\chi=\log\left\{\frac{\tan\frac{\varphi}{2}}{\tan\frac{\delta}{2}}\right\}, \quad \text{or } \tan\frac{\varphi}{2}=m\mathrm{e}^{l\chi} \quad . \quad . \quad . \quad (245).$$

This is the usual equation of the rumb line, and is identical with (234). Hence the polar spiral is a sort of curtated rumb line. If a rumb line be described on the surface of the sphere, its ordinate angle being $\left(\frac{\pi}{2} - \vartheta\right)$, and if we shorten its spherical radii vectores ϕ in the constant ratio given by the equation $\tan\theta = \tan\vartheta \sin\phi$, the extremity of θ will describe the polar spiral.

Another construction exhibiting the relation between those spirals may be given.

Let a concentric sphere be described, whose radius $\tan\vartheta = l$. On this sphere let a rumb line be constructed, having its pole in the axis of z. Let this rumb line be orthogonally projected on the tangent plane to the sphere, whose radius is 1, parallel to the plane of xy. Now if this plane curve be considered as the gnomonic projection (*i. e.* the eye being supposed at the centre) of a spherical curve, described on the surface of the outer sphere, this latter curve will be the polar spiral.



This we may thus show. In this construction we always have, $\tan\theta = \tan\vartheta \sin\phi$.

Now $CB = Qn$, $CB = \tan\theta$, and $Qn = \tan\vartheta \sin\phi$. Q and D are therefore the corresponding points of the rumb line and of the polar spiral, whose vector arcs are $CD = \theta$, $AQ = \phi$.

It is evident that the polar spiral has an asymptotic circle, whose radius is $\sin\vartheta$. In the vicinity of the pole, the polar spiral approximates indefinitely to the rumb line.

CXIX. To find the length of this spiral from the pole to the asymptotic circle.

$$\left(\frac{d\sigma}{d\phi}\right)^2 = \left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\left(\frac{d\chi}{d\phi}\right)^2 = \left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\left(\frac{d\chi}{dt}\right)^2\left(\frac{dt}{d\phi}\right)^2.$$

$$\tan\theta = l\sin\phi, \qquad \frac{d\theta}{d\phi} = \frac{l\cos\phi}{1 + l^2\sin^2\phi}, \qquad \frac{d\chi}{dt} = \varkappa, \ \frac{dt}{d\phi} = \frac{1}{\varkappa l\sin\phi},$$

and $\sin^2\theta = \dfrac{l^2 \sin^2\varphi}{1 + l^2 \sin^2\varphi}$. Introducing these relations,

we get $\dfrac{d\sigma}{d\varphi} = \dfrac{\sqrt{1+l^2}}{1 + l^2 \sin^2\varphi}$; dividing by $\cos^2\varphi$, and integrating,

we get $\quad \sigma = \tan^{-1}(\sqrt{1+l^2}\tan\varphi) \quad . \quad . \quad . \quad (246).$

When $\varphi = 0$, $\sigma = 0$, and when $\varphi = \dfrac{\pi}{2}$, $\sigma = \dfrac{\pi}{2}$.

We thus find that the length of the polar spiral between the pole and the asymptotic circle is equal to a quadrant of a great circle of a sphere,—a result in strict accordance with the more general theorem established in (CII).

When $\sqrt{1+l^2}\tan\varphi = 1$, or $\tan\varphi = \cos\vartheta$, $\sigma = \tan^{-1}(1)$ or $\sigma = \dfrac{\pi}{4}$.

CXX. To determine the velocity of the pole along the spiral.

As $\mathrm{v}^2 = \left(\dfrac{d\sigma}{dt}\right)^2 = \left(\dfrac{d\sigma}{d\varphi}\right)^2 \left(\dfrac{d\varphi}{dt}\right)^2 = \dfrac{(1+l^2)\ \varkappa^2\, l^2 \sin\varphi}{(1 + l^2\sin^2\varphi)^2}$;

$$\mathrm{v}^2 = \frac{\varkappa^2(1+l^2)\, l^2 \sin^2\varphi}{\{1 + l^2 \sin^2\varphi\}^2} = \frac{\varkappa^2\,(1+l^2)\tan^2\theta}{\sec^4\theta} = \varkappa^2(1+l^2)\sin^2\theta\cos^2\theta;$$

or $\mathrm{v} = \dfrac{\varkappa\sqrt{1+l^2}}{2}\sin 2\theta$, $\quad$ or $\mathrm{v} = \dfrac{1}{2}\ \dfrac{\varkappa \sin 2\theta}{\cos\vartheta}$.

CXXI. It was shown in (CXII.) that when k coincides with the greatest or the least principal axes of the body, the spirals described by the two other axes are equivalent to circular arcs. But when k coincides with b, the mean axe, the lengths of the spirals described by the greatest and the least principal axes are given by logarithms. Omitting the investigations, which, though somewhat complicated, the reader, assuming the principles established in the foregoing pages, may supply; the final result will be found to be

$i\,\sigma = j\log.\tan\dfrac{\psi}{2} + \log.(1 + j\sec\psi)$. i and j being constants.

SECTION VIII.

CXXII. WHEN the plane of the impressed moment coincides with the plane of one of the circular sections of the ellipsoid of moments, the elliptic integrals which determine the motion may be reduced from the *third* order to the *first*.

In this case $2k$ is the cyclic diameter of the ellipsoid, or the diameter perpendicular to the plane of one of its circular sections.

Accordingly $\frac{1}{k^2}=\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}$. Substitute this value of k in (169), (170), and (171). Reducing, we shall have $\psi+\varkappa t=$

$$\pm\left[\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}\right]\int\frac{d\phi}{\left[1-\frac{c^2}{a^2}\left(\frac{a^2-b^2}{b^2-c^2}\right)\sin^2\phi\right]\sqrt{1-\frac{c^4}{a^4}\left(\frac{a^2-b^2}{b^2-c^2}\right)^2\sin^2\phi}} \quad (247).$$

This integral, as the parameter is equal to the modulus, may be reduced to the first order as follows:

Let γ as in (XXIII.) be the *parametral angle* of the spherical parabola. Assume $\frac{1-\sin\gamma}{1+\sin\gamma}=\frac{c^2\,(a^2-b^2)}{a^2\,(b^2-c^2)}$. Whence

$$\sin\gamma=\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\frac{1}{c^2}-\frac{1}{a^2}},\ \cos^2\gamma=4\,\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{a}\right)^2} \quad (248).$$

The preceding equation may now be written

$$\psi+\varkappa t=\frac{2\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\left[1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} \quad (249).$$

If we compare this equation with (59), we shall find that the second member is equivalent to the following elliptic integral of the first order,

$$\sin\gamma \int \frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\} \quad (250).$$

The amplitudes ϕ and μ being connected by Lagrange's formula, $\tan(\phi-\mu)=\sin\gamma\tan\mu$, as in (60), or as it may in this case be written

$$\tan\phi = \frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\sin 2\mu}{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\cos 2\mu} \quad . \quad . \quad . \quad (251).$$

Should we require to reduce the integrals of the third and first order to the same amplitude, equation (63) will enable us with ease to do so, by assuming the theorem established in that equation,

$$\psi+\kappa t = \frac{\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} + \frac{1}{2}\tan^{-1}\left[\frac{\left(\frac{2\sin\gamma}{1+\sin\gamma}\right)\tan\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\right] \quad (252).$$

Hence ψ depends on an integral of the *first* order; the theorem it was proposed to establish.

CXXIII. Again, if we substitute the foregoing value of k in (155) which connects the time with the amplitude ϕ, on which immediately depends the position of the axis k in the body at the end of the given time, we shall have

$$\kappa t = \frac{1}{f^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}\int\frac{d\phi}{\sqrt{1-\frac{c^4}{a^4}\left(\frac{a^2-b^2}{b^2-c^2}\right)^2\sin^2\phi}};$$

and as $\dfrac{2\sin\gamma}{1+\sin\gamma} = \dfrac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}$

$$\frac{\left[\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right]}{2\left\{\frac{1}{a^2}+\frac{1}{c^2}-\frac{1}{b^2}\right\}}\kappa t = \frac{\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}} \quad (253).$$

But this elliptic integral, as shown in (XXXI.), is the expression for an arc of a spherical parabola, whose *parametral angle* is γ, the centre being the pole. In this case the two elliptic functions which determine the motion are represented by arcs of the *same* spherical parabola.

We may eliminate the latter integral by the equation established in (XXXI.), and the last equation will now become

$$\varkappa t = 2\,\frac{\left[\frac{1}{a^2}+\frac{1}{c^2}-\frac{1}{b^2}\right]}{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}\int\frac{d\mu}{\sqrt{1-\frac{4\,a^2\,c^2\,(a^2-b^2)\,(b^2-c^2)}{b^4\,(a^2-c^2)^2}\sin^2\mu}},$$

The moduli are two successive terms of Lagrange's modular scale.

SECTION IX.

CXXIV. In the foregoing investigations we found the lengths of the spirals, described by the greatest and least principal axes of the body, to be represented by elliptic integrals of the third order and logarithmic form. The normal geometrical representative of this integral will be found to be neither plane nor spherical, but parabolic. Indeed, this might have been anticipated; because the residual quantity, which in the second order is a right line, and in the third order and circular form an arc of a circle, in the form before us becomes a logarithm. Now, no plane curve on the surface of a sphere can be other than a circle. It was therefore a natural inquiry to seek for the required curve on a surface of revolution whose plane section should be expressed by a logarithm. Such is the common parabola. Accordingly we shall find, if we substitute for the sphere a paraboloid of revolution, and for the elliptic cone, which has its vertex at the centre of the sphere, an elliptic cylinder, that the curve of intersection of those surfaces may be rectified by an elliptic integral of the third order and logarithmic form.

Let the axes of the paraboloid of revolution and of the

elliptic cylinder coincide with the axis of z, the vertex of the paraboloid touching the plane of xy at the origin. Let k be the semi-parameter of the paraboloid, a and b the semi-axes of the ellipse in which the plane of xy intersects the cylinder. Then the equations of those surfaces, and consequently of the curve in which they intersect, are

$$x^2+y^2=2kz, \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad . \quad . \quad . \quad (254)$$

Let $d\Sigma$ be an element of the curve. We have by the elementary formula,

$$\frac{d\Sigma}{d\theta}=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2},$$

x, y, z being dependent functions of a fourth independent variable θ. Assume

$$x=a\cos\theta,\ y=b\sin\theta, \text{ then } a^2\cos^2\theta+b^2\sin^2\theta=2kz \ . \ (255)$$

differentiating and substituting

$$\left(\frac{d\Sigma}{d\theta}\right)^2=a^2\sin^2\theta+b^2\cos^2\theta+\frac{(a^2-b^2)^2}{k^2}\sin^2\theta\cos^2\theta \ . \ . \ (256)$$

To reduce this expression to a form suited for integration, it may be written

$$k^2\left(\frac{d\Sigma}{d\theta}\right)^2=b^2k^2+(a^2-b^2)\left[k^2+a^2-b^2\right]\sin^2\theta-(a^2-b^2)^2\sin^4\theta \ . \ (257)$$

$$\text{Let } \mathrm{P}=b^2k^2,\ \ \mathrm{Q}=(a^2-b^2)\left[k^2+a^2-b^2\right],\ \ \mathrm{R}=-(a^2-b^2)^2 \ . \ (258)$$

and the preceding equation will become, indicating the process of integration,

$$k\Sigma=\int\frac{d\theta\left[\mathrm{P}+\mathrm{Q}\sin^2\theta+\mathrm{R}\sin^4\theta\right]}{\sqrt{\mathrm{P}+\mathrm{Q}\sin^2\theta+\mathrm{R}\sin^4\theta}} \ . \ . \ . \ (259)$$

Let this trinomial be put as the product of two quadratic factors,

$$(\mathrm{A}+\mathrm{B}\sin^2\theta)(\mathrm{C}-\mathrm{B}\sin^2\theta), \text{ or } \mathrm{AC}+\mathrm{B}(\mathrm{C}-\mathrm{A})\sin^2\theta-\mathrm{B}^2\sin^4\theta. \quad (260)$$

Comparing this expression with the preceding in (257),

$$\mathrm{AC}=b^2k^2,\quad \mathrm{C}-\mathrm{A}=k^2+a^2-b^2,\quad \mathrm{B}=a^2-b^2. \ . \ . \ . \ (261)$$

To integrate this expression, Assume

$$\tan^2\varphi = \frac{A+B}{A}\tan^2\theta \quad . \quad . \quad . \quad (262)$$

The limits of integration of the complete functions will continue as before. Making the substitutions indicated by the preceding transformation, the integral will now become

$$\frac{\sqrt{C(A+B)}}{kb^2}\Sigma = \int \frac{d\varphi\left[1 - \frac{B}{C}\left(\frac{A+C}{A+B}\right)\sin^2\varphi\right]}{\left[1-\frac{B}{A+B}\sin^2\varphi\right]^2\sqrt{1-\frac{B}{C}\left(\frac{A+C}{A+B}\right)\sin^2\varphi}} \, . \, (263)$$

Let $\frac{B}{A+B} = n$, $\frac{B}{C}\left(\frac{A+C}{A+B}\right) = c^2$, therefore $\frac{c^2}{n} = \frac{A+C}{C}$ . . (264)

It will presently be shown that $A+C$ is always greater than C; whence $c^2 > n$.

As $c^2 = \dfrac{1+\frac{A}{C}}{1+\frac{A}{B}}$, and as B will be shown to be always less than C, c^2 must always be less than 1. Therefore, as c^2 is less than 1 and greater than n the parameter with a *negative* sign; the elliptic integral, to which the above *hyperlogarithmic* expression may be reduced, must be of the *third* order and *logarithmic* form.

CXXV. To reduce the hyperlogarithmic integral (263) to the usual form of an elliptic integral.

Assume $\Phi = \sin\varphi\cos\varphi\,(1-c^2\sin^2\varphi)^{\frac{1}{2}}\,(1-n\sin^2\varphi)^{-1}$ . . (265)

Differentiate this expression with respect to φ, and we shall have

$$\frac{d\Phi}{d\varphi} = \frac{1-2(1+c^2)\sin^2\phi+3c^2\sin^4\phi}{[1-n\sin^2\phi]\sqrt{1-c^2\sin^2\phi}} + \frac{2n(\sin^2\phi-\sin^4\phi)\,(1-c^2\sin^2\phi)}{[1-n\sin^2\phi]^2\sqrt{1-c^2\sin^2\phi}} . \quad (266)$$

Let $(1-n\sin^2\varphi) = N$, $1-c^2\sin^2\varphi = \Delta^2 = \varpi$.

Separating the numerators of the preceding expression into their component parts, and attaching their respective denominators to each, we shall have

$$\frac{1}{N\Delta} = \frac{1}{N\Delta} \quad . \quad . \quad . \quad . \quad (a)$$

$$\frac{-2(1+c^2)\sin^2\phi}{N\Delta} = \frac{2(1+c^2)}{n}\frac{(1-n\sin^2\phi-1)}{N\Delta} =$$

$$\frac{2(1+c^2)}{n\Delta} - \frac{2(1+c^2)}{nN\Delta} \quad . \quad . \quad . \quad (b).$$

$$\frac{3c^2\sin^4\phi}{N\Delta} = -\frac{3c^2}{n}\frac{(1-n\sin^2\phi-1)\sin^2\phi}{N\Delta} = -\frac{3c^2}{n}\frac{\sin^2\phi}{\Delta} + \frac{3c^2\sin^2\phi}{nN\Delta},$$

and $$-\frac{3c^2}{n}\frac{\sin^2\phi}{\Delta} = \frac{3}{n}\frac{(1-c^2\sin^2\phi-1)}{\Delta} = \frac{3\Delta}{n} - \frac{3}{n\Delta};$$

but $$\frac{3c^2\sin^2\phi}{nN\Delta} = -\frac{3c^2}{n^2}\frac{(1-n\sin^2\phi-1)}{N\Delta} = -\frac{3c^2}{n^2\Delta} + \frac{3c^2}{n^2N\Delta};$$

whence, $$\frac{3c^2\sin^4\phi}{N\Delta} = \frac{3\Delta}{n} - \frac{3}{n\Delta} - \frac{3c^2}{n^2\Delta} + \frac{3c^2}{n^2N\Delta} \quad . \quad . \quad . \quad (c).$$

Combining the expressions in (a), (b), and (c),

$$\{1-2(1+c^2)\sin^2\phi+3c^2\sin^4\phi\}(1-n\sin^2\phi)^{-1}(1-c^2\sin^2\phi)^{-\frac{1}{2}} =$$

$$\frac{3\Delta}{n} + \left\{\frac{2}{n}(1+c^2) - \frac{3c^2}{n} - \frac{3}{n}\right\}\frac{1}{\Delta} + \left\{1 - \frac{2}{n}(1+c^2) + \frac{3}{n^2}c^2\right\}\frac{1}{N\Delta} \cdot (d).$$

The second term $\dfrac{2n(\sin^2\phi-\sin^4\phi)\varpi}{N^2\Delta}$ in (266), may be thus reduced.

$$-\frac{2n\varpi\sin^4\phi}{N^2\Delta} =$$

$$-\frac{2n\varpi}{n^2}\left\{\frac{1-2n\sin^2\phi+n^2\sin^4\phi-2+2n\sin^2\phi+1}{N^2\Delta}\right\} =$$

$$-\frac{2n\varpi}{n^2\Delta} + \frac{4n\varpi}{n^2N\Delta} - \frac{2n\varpi}{n^2N^2\Delta}.$$

Now $$-\frac{2n\varpi}{n^2\Delta} = -\frac{2\Delta}{n}, \text{ and } \frac{4n\varpi}{n^2N\Delta} = \frac{4(1-c^2\sin^2\phi)}{nN\Delta} =$$

$$\frac{4}{nN\Delta} + \frac{4c^2}{n^2}\frac{(1-n\sin^2\phi-1)}{N\Delta} = \frac{4c^2}{n^2\Delta} - \frac{4(c^2-n)}{n^2N\Delta}.$$

Combining those results

$$-\frac{2n\varpi\sin^4\phi}{N^2\Delta}=\frac{-2\Delta}{n}+\frac{4c^2}{n^2}\frac{1}{\Delta}-\frac{4}{n^2}(c^2-n)\frac{1}{N\Delta}-\frac{2\varpi}{nN^2\Delta}\quad . \text{ (e).}$$

In like manner,

$$\frac{2n\varpi\sin^2\phi}{N^2\Delta}=-\frac{2n\varpi}{n}\left\{\frac{1-n\sin^2\phi-1}{N^2\Delta}\right\}=-\frac{2\varpi}{N\Delta}+\frac{2\varpi}{N^2\Delta};$$

but

$$-\frac{2\varpi}{N\Delta}=-\frac{2(1-c^2\sin^2\phi)}{N\Delta}=-\frac{2}{N\Delta}-\frac{2c^2}{n}\frac{(1-n\sin^2\phi-1)}{N\Delta},$$

and this latter $=-\dfrac{2c^2}{n\Delta}+\dfrac{2c^2}{nN\Delta}$; whence

$$\frac{2n\varpi\sin^2\phi}{N^2\Delta}=-\frac{2c^2}{n\Delta}-2\left(1-\frac{c^2}{n}\right)\frac{1}{N\Delta}+\frac{2\varpi}{N^2\Delta}\quad . \quad . \quad . \text{ (f).}$$

Adding (e) and (f) together,

$$\frac{2n\varpi(\sin^2\phi-\sin^4\phi)}{N^2\Delta}=$$

$$-\frac{2\Delta}{n}+\left\{\frac{4c^2}{n^2}-\frac{2c^2}{n}\right\}\frac{1}{\Delta}+\left\{\frac{2c^2}{n}-2+\frac{4}{n}-\frac{4c^2}{n^2}\right\}\frac{1}{N\Delta}-2\left(\frac{1}{n}-1\right)\frac{\varpi}{N^2\Delta}. \text{ (g).}$$

Adding (d) and (g) together, we get finally,

$$\frac{d\Phi}{d\phi}=\frac{\Delta}{n}+\frac{1}{n}\left(\frac{c^2}{n}-1\right)\frac{1}{\Delta}+\left[\left(\frac{1}{n}-1\right)-\left(\frac{c^2}{n^2}-\frac{1}{n}\right)\right]\frac{1}{N\Delta}-2\left(\frac{1}{n}-1\right)\frac{\varpi}{N^2\Delta}\quad . \text{ (h).}$$

Transposing, dividing, substituting, writing n for $\dfrac{B}{A+B}$, and c^2 for $\dfrac{B(A+C)}{C(A+B)}$, integrating the above expression, and introducing the relations found in (263),

$$\frac{2\sqrt{C(A+B)}}{kb^2}\Sigma=2\int\frac{(1-c^2\sin^2\phi)\,d\phi}{[1-n\sin^2\phi]^2\sqrt{1-c^2\sin^2\phi}},$$

we obtain the final result,

$$2k\sqrt{C(A+B)}.\,z = -BC\Phi + A(C-A-B)\int\frac{d\phi}{[1-n\sin^2\phi]\sqrt{1-c^2\sin^2\phi}}$$

$$+A(A+B)\int\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}} \quad + C(A+B)\int d\phi\sqrt{1-c^2\sin^2\phi} \quad (267).$$

CXXVI. We shall now develop the remarkable relations which exist between the constants of this equation and the constants k, a, b, of the given surfaces. The modulus, parameter, coefficients, and criterion of circularity, may be represented as linear products of constants, having simple relations with those of the given surfaces.

Resuming the equations given in (261),

$$\text{AC} = b^2k^2, \quad \text{C}-\text{A} = k^2+a^2-b^2, \quad \text{B} = a^2-b^2.$$

We find $\quad (\text{A}+\text{C})^2 = (k^2+a^2-b^2)^2 + 4b^2k^2.$

Assume $\;4p^2 = k^2+(a+b)^2$, and $4q^2 = k^2+(a-b)^2$ (268);

we then shall have the following relations,

$$\left.\begin{array}{ll}
\text{A}+\text{C}=4pq, & \text{B}=(a+b)(a-b), \\
\text{A}+\text{B}=(a+p-q)(a+q-p), & \text{C}-\text{B}=(p+q+a)(p+q-a), \\
\text{A}=(b+p-q)(b+q-p), & k^2+a^2+b^2=2(p^2+q^2), \\
\text{C}=(p+q+b)(p+q-b), & ab=(p+q)(p-q).
\end{array}\right\} \quad (269).$$

$$c^2 = \frac{4(a+b)(a-b)pq}{(p+q+b)(p+q-b)(a+p-q)(a+q-p)} \quad \ldots \quad (270)$$

$$n = \frac{(a+b)(a-b)}{(a+p-q)(a+q-p)} \quad \ldots \quad (271);$$

and if we denote by $\varkappa$ the criterion of circularity,

$$\varkappa = \frac{-b^4}{a^2(p+q)^2}\left(\frac{p+q+a}{p+q+b}\right)^2\left(\frac{p+q-a}{p+q-b}\right)^2, \quad . \quad (272)$$

a quantity essentially negative.

As $\text{AC} = b^2k^2$, $\quad$ and $(\text{A}+\text{B})(\text{C}-\text{B}) = a^2k^2$.

$$a^2b^2k^4 = \text{AC}(\text{A}+\text{B})(\text{C}-\text{B}).$$

Substituting, we shall obtain the value of k^4 as a product of twelve symmetrical linear factors, as follows:

$$\left.\begin{aligned} k^4 = &(a+p+q)(a+p-q)(a+q-p)(p+q-a)(p+q)^{-1}a^{-1} \\ &\times(b+p+q)(b+p-q)(b+q-p)(p+q-b)(p-q)^{-1}b^{-1} \end{aligned}\right\} (273);$$

an expression of remarkable symmetry for the semi-parameter of the paraboloid.

As a numerical example, let $a=12$, $b=7$, $p=10$, $q=4$. We shall find $k=\sqrt{39}$.

To express a and b, the semi-axes of the base of the elliptic cylinder, as functions of the modulus and parameter of the logarithmic integral and k, the semi-parameter of the paraboloid, that is, as functions of k, n and c. We shall have, eliminating A, B, C from the equations,

$$n=\frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}}, \quad c^2=\frac{\mathrm{B}(\mathrm{A}+\mathrm{C})}{\mathrm{C}(\mathrm{A}+\mathrm{B})}, \quad \mathrm{C}-\mathrm{A}=k^2+a^2-b^2, \quad \mathrm{B}=a^2-b^2,$$

$$\frac{a^2}{k^2}=\frac{n(c^2-n)(1-c^2)}{[2n-c^2-n^2]^2}, \qquad \frac{b^2}{k^2}=\frac{n(c^2-n)(1-n)^2}{[2n-c^2-n^2]^2} \quad \ldots \quad (274)$$

In order that the values of a and b may be real, we must have n positive, $c^2>n$, and $1>c^2$. Hence, as the type of the logarithmic form of the elliptic integral of the third order is

$$\int\frac{d\phi}{[1-i^2\sin^2\phi]\sqrt{1-(i^2+j^2)\sin^2\phi}}, \quad (i^2+j^2) \text{ being less than } 1,$$

we may therefore name the curve in which a paraboloid of revolution is intersected by an elliptic cylinder, whose axis coincides with that of the paraboloid, as the *logarithmic ellipse.*

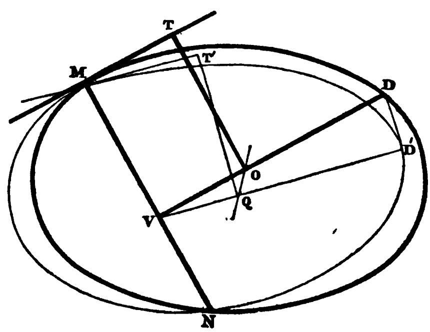
It is obvious that we cannot, as in the circular form, reduce the order of the integral from the third to the first, by making, as in (46) $c=n$; for this would render the values of a and b imaginary. We cannot, therefore, by assuming a particular value for the parameter, reduce the logarithmic form of the third order to an elliptic integral of the first order.

CXXVII. In the preceding investigations the element of the curve has been taken as a side of a limiting rectilinear polygon inscribed within it. We may, however, effect the rectification of

the curve, starting from other elementary principles. For this purpose, we may conceive the element of the *logarithmic ellipse* as identical with the coinciding element of a plane ellipse whose axes vary from point to point along the logarithmic ellipse. If we then equate together the two different expressions thus found for the same arc, we shall obtain the equations given by Legendre for the comparison of integrals of the *logarithmic* form, in the same way as two different expressions for the same arc of a spherical ellipse gave us the forms of comparison for elliptic integrals of the third order and *circular* form.

As it is not possible that a sphere, a paraboloid of revolution, and an elliptic cylinder should intersect mutually in a common curve, the reader will not fail to observe the geometrical reason why the most eminent analysts have failed in the attempt to reduce the two forms, which are classed under the same order, one to the other. In fact, they are plainly irreducible.

Let M be a point on the logarithmic ellipse. Through M let a plane be drawn perpendicular to the axis of the cylinder, or to the axis of Z. It will cut the cylinder in a plane ellipse, whose semi-axes are a and b; this is evident. Through M draw the tangent MT to the plane ellipse and the normal MN. Through the point



M let MT′ be drawn a tangent to the logarithmic ellipse, and through the straight lines MN, MT′ let a plane be drawn; it will intersect the cylinder in a plane ellipse, whose plane will cut the plane of the former in the common normal MN, and is inclined to it at the angle $\text{TMT}' = \mu$. Now this variable plane ellipse, and the logarithmic ellipse on the surface of the paraboloid, may be taken as mutually osculating curves. The plane ellipse, which varies from point to point along the logarithmic ellipse, has the same analogy to it as the circle of curvature has to a plane curve. To apply this reasoning.

Through the axis of the cylinder let a plane be drawn parallel

to the tangent plane TMT′, and cutting the normal in the point V, the plane of the *constant* ellipse in the line VD, and the plane of the *variable* ellipse in the line VD′. As MV is perpendicular to this diametral plane of the cylinder passing through the axis, it is perpendicular to VD and to VD′; therefore the angle DVD′ $= \mu$.

As all the plane sections of this elliptic cylinder will have their centres on its axis OQ, the points O and Q, in which the planes of the constant and variable ellipses intersect it, are the centres of those sections; and if we join the point M with O and Q respectively, the lines OD and QD′ will be the conjugate diameters of those lines. From O let the perpendicular OT be drawn $=$ P, and from Q the perpendicular QT′ $=$ P′. Let OD $= r$, then QD′ $= r \sec \mu$.

Now the area of the constant ellipse, being the product of the perpendicular from the centre on the tangent, and the conjugate diameter to this tangent, is $= r \mathrm{P} \pi$; and the area of the variable ellipse $= r' \mathrm{P}' \pi = r \mathrm{P}' \pi \sec \mu$. But the area of the constant ellipse is the projection of the area of the variable ellipse. Therefore $\mathrm{P} r \pi = \{\mathrm{P}' r \pi \sec \mu\} \cos \mu$,

$$\text{or } \mathrm{P}' = \mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad (275).$$

In (30) and (31) we found for the element of an arc of a plane curve the expression,

$\dfrac{ds}{d\lambda} = p + \dfrac{d^2 p}{d\lambda^2}$; and as $\mathrm{P}'^2 = \mathrm{P}^2 = a^2 \cos^2 \lambda + b^2 \sin^2 \lambda$,

$$\frac{d^2 \mathrm{P}'}{d\lambda^2} = -\frac{(a^2 - b^2)\{a^2 \cos^4 \lambda - b^2 \sin^4 \lambda\}}{(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)^{\frac{3}{2}}} \quad . \quad . \quad (276);$$

$$\text{or, } \frac{d\Sigma}{d\lambda} = (a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)^{\frac{1}{2}} - \frac{(a^2 - b^2)(a^2 \cos^4 \lambda - b^2 \sin^4 \lambda)}{(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)^{\frac{3}{2}}} \quad (277).$$

Before we proceed to integrate this expression, we must observe that $d\lambda$ being the element of the angle between two consecutive normals in the plane ellipse, it is the projection of the element of the angle between two consecutive normals in the variable ellipse; this element must therefore be $d\lambda \sec \mu$. We are now in a position to apply the formula for plane rectification given in (30). We have accordingly

$$\Sigma' = \int d\lambda \left[\frac{a^2 \cos^2\lambda + b^2 \sin^2\lambda}{\cos\mu}\right]^{\frac{1}{2}} - (a^2 - b^2)\int \frac{d\lambda\,(a^2\cos^4\lambda - b^2\sin^4\lambda)}{\cos\mu\,(a^2\cos^2\lambda + b^2\sin^2\lambda)^{\frac{3}{2}}} \quad (278).$$

We have now to express $\cos\mu$ in terms of λ.

As $\sec^2\mu = \frac{d\Sigma^2}{dx^2 + dy^2} = \frac{b^2k^2 + (a^2-b^2)(k^2+a^2-b^2)\sin^2\theta - (a^2-b^2)^2\sin^4\theta.}{k^2(a^2\sin^2\theta + b^2\cos^2\theta)}$

Eliminating $\frac{y}{x}$ between the equations $\tan\lambda = \frac{a^2}{b^2}\frac{y}{x}$,

and $\frac{y}{x} = \frac{b}{a}\tan\theta$, we shall have $\tan\lambda = \frac{a}{b}\tan\theta$ . . (278*).

Eliminating $\tan\theta$ by this equation, we obtain

$$\cos^2\mu = \frac{k^2(a^2\cos^2\lambda + b^2\sin^2\lambda)}{a^2k^2 + (a^2-b^2)(a^2-b^2-k^2)\sin^2\lambda - (a^2-b^2)^2\sin^4\lambda} \quad (279).$$

We may, as in (258), write P′, Q′, R′ for the coefficients of $\sin\lambda$ in this equation, and the resulting expression will become

$$k\Sigma' = \int d\lambda\,\sqrt{P' + Q'\sin^2\lambda + R'\sin^4\lambda} - (a^2-b^2)\int \frac{d\lambda\,(a^2\cos^4\lambda - b^2\sin^4\lambda)}{k\cos\mu\,(a^2\cos\lambda + b^2\sin\lambda)^{\frac{3}{2}}} \quad (280).$$

As the first of those integrals is precisely similar in form to the one obtained in (259), we may in the same manner reduce it into factors, writing α, β, γ instead of A, B, C. Accordingly let

$$P' + Q'\sin^2\lambda + R'\sin^4\lambda = (\alpha + \beta\sin^2\lambda)(\gamma - \beta\sin^2\lambda), \quad (281)$$

following the investigation in (CXXV.), step by step, we shall have, as in (262) and (264), ψ, m, c', being the amplitude, parameter, and modulus,

$$\tan^2\psi = \frac{\alpha+\beta}{\alpha}\tan^2\lambda; \quad m = \frac{\beta}{\alpha+\beta}; \quad c'^2 = \frac{\beta(\alpha+\gamma)}{\gamma(\alpha+\beta)} \;. \quad (282).$$

As $\alpha\gamma = a^2k^2$, $\beta = a^2 - b^2$, and $\gamma - \alpha = a^2 - b^2 - k^2$,

we shall have the following relations between the constants in (260) and (281), α, β, γ, m, c'^2, ψ, and A, B, C, n, c^2, ϕ.

$$\left.\begin{array}{l}\beta=\text{B}, \qquad \alpha=\text{C}-\text{B}, \qquad \gamma=\text{A}+\text{B}, \\ \alpha+\gamma=\text{A}+\text{C}, \qquad \gamma-\beta=\text{A}, \qquad \alpha+\beta=\text{C}. \\ \gamma-\alpha-\beta+\text{C}-\text{A}-\text{B}=0. \\ c'^2=\dfrac{\beta(\alpha+\gamma)}{\gamma(\alpha+\beta)}=\dfrac{\text{B}\,(\text{A}+\text{C})}{(\text{A}+\text{B})\,\text{C}}=c^2, \text{ therefore } c'=c^2. \\ m=\dfrac{\beta}{\alpha+\beta}=\dfrac{\text{B}}{\text{C}}. \end{array}\right\} (283).$$

The moduli c and c' are therefore the same in the two functions, and the parameters m and n are connected by the equation

$$m+n-mn=c^2.$$

The amplitudes φ and ψ are equal.

In (262) we assumed $\tan^2\varphi=\dfrac{\text{A}+\text{B}}{\text{A}}\tan^2\theta$; and in (282),

$$\tan^2\psi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda, \qquad \text{but } \tan^2\lambda=\frac{a^2}{b^2}\tan^2\theta, \text{ as in } (278^*).$$

Whence $$\tan^2\psi=\frac{a^2}{b^2}\,\frac{(\alpha+\beta)}{(\text{A}+\text{B})}\,\frac{\text{A}}{\alpha}\tan^2\varphi.$$

In (283) it was shown that $\alpha+\beta=\text{C}$, $\alpha=\text{C}-\text{B}$, whence

$$\tan^2\psi=\frac{a^2}{b^2}\,\frac{\text{A}\,\text{C}\tan^2\varphi}{(\text{A}+\text{B})(\text{C}-\text{B})}.$$

Now $\text{A}\,\text{C}=b^2k^2$, and $(\text{A}+\text{B})(\text{C}-\text{B})=a^2k^2$;

We hence deduce $$\tan^2\psi=\tan^2\varphi, \quad \text{or } \psi=\varphi. \qquad (284).$$

CXXVIII. We shall now proceed to find the value of the second integral in (280).

From (289) we deduce $$\tan^2\mu=\frac{(a^2-b^2)^2\sin^2\lambda\cos^2\lambda}{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}. \quad (285).$$

Differentiating this expression, reducing, dividing by cos μ, and integrating, we get the following equation,

$$k\int\frac{d\mu}{\cos^3\mu}=(a^2-b^2)\int\frac{d\lambda\,(a^2\cos^4\lambda-b^2\sin^4\lambda)}{\cos\mu\,(a^2\cos^2\lambda+b^2\sin\lambda)^{\frac{3}{2}}} \qquad (285^*).$$

(290) may now be written

$$k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\mu}{\cos^3\mu} \quad . \qquad (286).$$

If we measure the arc of the logarithmic ellipse from the minor principal arc, or from the parabolic arc which is projected into b, instead of placing the origin at the vertex of the major axe as in (255); we must in this case put $x=a\sin\vartheta$, $y=b\cos\vartheta$, and following the steps there indicated, we shall at length obtain

$$k\,S=\int d\vartheta\sqrt{P'+Q'\sin^2\vartheta+R'\sin^4\vartheta}. \quad . \quad . \qquad (287).$$

If we now make $\vartheta=\lambda$, and subtract the two latter equations one from the other, the resulting equation will become

$$S-\Sigma=k\int\frac{d\mu}{\cos^3\mu} \quad . \quad . \quad . \quad . \quad . \qquad (288).$$

But this integral is, we know, the expression for an arc of a parabola whose semi-parameter is k, measured from its vertex to a point on the curve where its tangent makes the angle μ with the ordinate.

Thus the difference between two elliptic arcs measured from the vertices of the curve, which in the plane ellipse may, as we know, be expressed by a right line, and in the spherical ellipse by an arc of a circle*, as we have elsewhere shown, will be represented in the logarithmic ellipse by an arc of a parabola. As a parabolic arc may be rectified by a logarithm, we hence see the propriety of the term *logarithmic* as applied to this function.

CXXIX. Let us resume the equation (286),

$$k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\mu}{\cos^3\mu}.$$

* Philosophical Magazine, July 1844.

We shall now proceed to develop the first integral of the second side of this equation. As the first member of this integral is precisely the same in form as (259), and the amplitude $\psi=\phi$, we may substitute α, β, γ for A, B, C, m for n, Φ' for Φ, retaining the modulus which continues unchanged, and write

$$2k\sqrt{\gamma(\alpha+\beta)}\Sigma=-\beta\gamma\Phi'+\alpha(\gamma-\alpha-\beta)\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-c^2\sin^2\phi}}$$

$$+\alpha(\alpha+\beta)\int\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}}+\gamma(\alpha+\beta)\int d\phi\sqrt{1-c^2\sin^2\phi}-2k^2\sqrt{\gamma(\alpha+\beta)}\int\frac{d\mu}{\cos^2\mu} \quad (290)$$

Or, if we assume the relations given in (283) between α, β, γ and A, B, C, we may express the former in terms of the latter, and write the preceding equation in the form

$$2k\sqrt{C(A+B)}\Sigma=-B(A+B)\Phi'-(C-A-B)(C-B)\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-c^2\sin\phi}}$$

$$+C(C-B)\int\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}}+C(A+B)\int d\phi\sqrt{1-c\sin^2\phi}-2k^2\sqrt{C(A+B)}\int\frac{d\mu}{\cos^2\mu} \quad (291).$$

If we compare together (267) and (291), which are expressions for the same arc of the logarithmic ellipse, and make the obvious reductions, introducing m and n, the parameters through their values, $n=\frac{B}{A+B}$, $m=\frac{B}{C}$, and putting for Φ and Φ' their values $\frac{\sin\phi\cos\phi\sqrt{1-c^2\sin^2\phi}}{(1-n\sin^2\phi)}$, and $\frac{\sin\phi\cos\phi\sqrt{1-c^2\sin^2\phi}}{(1-m\sin^2\phi)}$, we shall get the resulting equation of comparison,

$$\left(\frac{1-n}{n}\right)\int\frac{d\phi}{[1-n\sin^2\phi]\sqrt{1-c^2\sin^2\phi}}+\left(\frac{1-m}{m}\right)\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-c^2\sin^2\phi}}$$

$$=\frac{c^2}{mn}\int\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}}+\frac{\sin\phi\cos\phi\sqrt{1-c^2\sin^2\phi}}{(1-n\sin^2\phi)(1-m\sin^2\phi)}-\frac{2}{\sqrt{mn}}\int\frac{d\mu}{\cos^2\mu} \quad . \quad (292).$$

From (285) we may deduce $\sin\mu=\sqrt{mn}\,\frac{\sin\phi\cos\phi}{\sqrt{1-c^2\sin^2\phi}}$, we shall therefore have

$$\frac{1}{\sqrt{mn}}\,\frac{\sin\mu}{\cos^2\mu}=\frac{\sin\phi\cos\phi\sqrt{1-c^2\sin\phi}}{(1-n\sin^2\phi)(1-m\sin^2\phi)}.$$

It was shown in (XXIV.), that $\frac{\sin \mu}{\cos^2 \mu}$ represents the portion of a tangent to a parabola intercepted between the point of contact and the perpendicular from the focus. We have therefore,

$$\frac{\sin \mu}{\cos^2 \mu} = 2\int \frac{d\mu}{\cos^3 \mu} - \int \frac{d\mu}{\cos \mu}. \quad . \quad . \quad (293)$$

Substituting this value of $\frac{\sin \mu}{\cos^2 \mu}$ in the preceding equation, we get, using the ordinary notation of elliptic functions,

$$\left(\frac{1-n}{n}\right)\Pi_c(n,\varphi) + \left(\frac{1-m}{m}\right)\Pi_c(m,\varphi) = \frac{c^2}{mn}\mathrm{F}_c(\varphi) - \frac{1}{\sqrt{mn}}\int \frac{d\mu}{\cos \mu};$$

which is precisely the formula established by Legendre, *Traité des Fonctions Elliptiques,* tom. i. p. 68.

CXXX. We may express a and b the semi-axes of the elliptic base of the cylinder in terms of m and n, the parameters of the two elliptic functions in the preceding equation. From the equation of condition $c^2 = m + n - mn$, and (274), eliminating c^2 we get,

$$\frac{a^2}{k^2} = \frac{nm(1-m)}{(n-m)^2}, \qquad \frac{b^2}{k^2} = \frac{mn(1-n)}{(n-m)^2}. \quad . \quad (294).$$

We may perceive that n and m cannot be equal in the *logarithmic* integral, as they may in the *circular* integral.

If we put $\varkappa'$ for the criterion of circularity derived from the parameter m instead of n, as in (272), and multiply the two together, we shall get for $\varkappa\varkappa'$ a perfect fourth power.

$$\varkappa\varkappa' = \left(\frac{b}{a}\right)^4 \left\{\frac{p+q+a}{p+q+b}\right\}^4 \left\{\frac{p+q-a}{p+q-b}\right\}^4 \quad . \quad . \quad (295).$$

If we define the trigonometrical tangent of an arc of a conic section as the inverse ratio of the semi-parameter to the portion of the linear tangent between the vertex and the diameter passing through the extremity of the arc, we may exhibit the polar equation of the logarithmic ellipse in a form analogous to (2). If we denote this new species of tangent by the symbol $\tau\alpha\nu.$ to distinguish it from the circular tan., we shall have in the parabola, $\tau\alpha\nu.\ \sigma = \frac{y}{k}.$

Let α and β be the greatest and least principal parabolic arcs

of the logarithmic ellipse,—that is, the arcs drawn from the vertex of the paraboloid to the vertices of the curve, and ρ the parabolic arc drawn to any point on the curve, making the angle ψ with α. We may show that

$$\frac{1}{\tau\alpha\nu.^2\,\rho}=\frac{\cos^2\psi}{\tau\alpha\nu.^2\,\alpha}+\frac{\sin^2\psi}{\tau\alpha\nu.^2\,\beta}. \quad . \quad . \quad . \quad . \quad (296).$$

CXXXI. We may also show that a plane ellipse, a spherical parabola, a spherical ellipse, and a logarithmic ellipse, may all four be described on the *same* elliptic cylinder. These curves are the intersections of the cylinder by a plane, two spheres, and a paraboloid: one of the spheres and the paraboloid may be of arbitrary magnitude. The radius R of the sphere which gives the spherical parabola is constant.

This may be shown as follows:

In order that a section of a sphere by a cylinder may be a spherical parabola, a certain relation must exist between its principal arcs α and β. Let γ be the *parametral* angle of the spherical parabola, and R the radius of the required sphere.

We may derive from (56) the following values.

$$\sin^2\alpha=\frac{1+\sin\gamma}{2}=\frac{a^2}{\mathrm{R}^2}, \qquad \sin^2\beta=\frac{2\sin\gamma}{1+\sin\gamma}=\frac{b^2}{\mathrm{R}^2}. \; (297).$$

Hence $\sin\gamma=\dfrac{a^2 b^2}{\mathrm{R}^4}$, or $\sqrt{\sin\gamma}$ is the ratio of the area of the projection of the spherical parabola to the area of a great circle of the sphere.

Eliminating $\sin\gamma$ from (307), $\mathrm{R}=a\sqrt{1+e}$; $2a$ being the major axe, and e the eccentricity of the base of the elliptic cylinder.

Hence all elliptic integrals will have common properties, especially such as are projective. For example, they will all four be complete between the same limits 0 and $\frac{\pi}{2}$; the substitutions in one form will often serve for all; the curves will have points analogous to centres and foci, &c.

CXXXII. The preceding investigations lead us to consider a new classification of elliptic integrals, which, in a geometrical point of view, would seem to be more natural than the one at present in use. As the first order is merely a particular case of the

circular form of the third; its geometrical type—the spherical parabola—being a particular species of spherical conic, while the two forms which are classed under the third order, are irreducible, one to the other, representing, as they do, curves of different species, it would seem a more appropriate division, to found their classification on their geometrical types, the *plane*, the *logarithmic*, and the *spherical* ellipses, which those integrals represent. Thus, that which is now the second, would stand the first; the logarithmic form of the third order would hold the second place; while the circular form of the third order, of which the present first order is a particular case, would occupy the third rank. However, as the present division has been sanctioned by time, and by the great names of the founders of this department of mathematical science, Legendre, Jacobi, Abel, and others, it would be presumptuous to propose to change it. Besides, in a point of view purely analytical,—the view of the inventors,—the present division of these integrals may be held to be the most appropriate; for example, it naturally presents itself in the computation of tables of the numerical values of those integrals.

If a right cylinder, standing on a plane hyperbola as a base, be substituted for the elliptic cylinder, the curve of intersection with the paraboloid may be named the *logarithmic hyperbola.* It will have four infinite branches, whose asymptotes will be the infinite arcs of two equal plane parabolas. This curve, and not the spherical ellipse, is the true analogue of the common hyperbola. We may, for example, express the difference between the infinite arcs of the logarithmic hyperbola and the asymptotic parabola, by a logarithmic integral. The curved asymptotic spaces may be represented by a plane hyperbola, and by circular integrals. But as these, and many other kindred properties of this curve, have no immediate bearing on the subject of this essay, their investigation must be here omitted.

THE END.

London:
Spottiswoodes and Shaw,
New-street-Square.

By the same Author,

And Sold by George Bell, Fleet Street.

I.

Tangential Co-ordinates;

Or, the Application of a New Analytical Method to the Theory of Curves and Curved Surfaces. 8vo.

II.

A Short Treatise on Spherical Conic Sections.

8vo.

III.

On a New Method of Converting Rectilinear

into Rotatory Motion. 8vo.

IV.

Education and Educational Institutions

Considered, with reference to the Industrial Professions and the present Aspect of Society. 8vo.

V.

A Sermon,

On the Death of the Rev. T. Tattershall, D.D., Vice-President of the Literary and Philosophical Society of Liverpool, preached in the Church of St. Augustine, Liverpool, November the First, 1846.

VI.

Examination the Province of the State;

Or, the Outlines of a Practical System for the Extension of National Education.